

ON INFINITE SETS WITH NO 3 ON A LINE

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ABSTRACT. We give a construction of an infinite set of points A in \mathbb{R}^2 such that any subset $P \subseteq A$ has a constant density subset P' with no three points collinear and yet A cannot be separated into finitely many subsets such that each subset has no three points collinear. This provides a new proof of a question of Erdős, Nešetřil, and Rödl. The construction was generated by an internal model at OpenAI.

1. INTRODUCTION

Theorem 1.1. *There exists an infinite set $A \subseteq \mathbb{R}^2$ with the following properties:*

- *For any subset P of n points, there exists a subset P' of size at least $n/2$ such that no three points in P' are collinear.*
- *A cannot be split into finitely many sets $A_1 \cup \dots \cup A_m$ such that each A_i contains no three collinear points.*

This resolves a question of Erdős, Nešetřil, and Rödl [1]. After sharing our proof of Theorem 1.1 with Rödl, he noted in personal communication that Theorem 1.1 can be deduced from work of [3, Theorem 1.7] (noting that the only collinear triples in $[3]^n$ correspond to 3-term arithmetic progressions and then a generic projection provides the result for Theorem 1.1). Due to the comparative simplicity of our proof, we record it here.

We further remark that one can pose a finitary version of the question by considering a set A of k points such that every subset satisfies the first item in Theorem 1.1, and asking how many sets A_i are required to partition A into pieces such that each piece has no three on a line. The construction here gives that $\Omega(\log k / \log \log k)$ sets may be required; determining the true dependence appears to be an interesting question. Note that an $O(\log k)$ upper bound follows in the finitary question via iteratively applying the first condition.

This theorem can be viewed as a geometric question of “Pisier” type. We refer the reader to Nešetřil, Rödl and Sales [2] and Reiher, Rödl and Sales [3] for context, background and recent results regarding such questions.

1.1. Comment on use of AI. The construction and proof were generated by a model internal to OpenAI. The human authors digested the proof and have presented it in a human readable form (and modified it for clarity and elegance).

2. PROOF OF THEOREM 1.1

Let $(t_i)_{i \in \mathbb{N}}$ be a sequence of real numbers that are algebraically independent.¹ We now construct A to be the set of points indexed by $(i, j) \in \binom{\mathbb{N}}{2}$ such that $(i, j) \rightarrow (t_i + t_j, t_i^2 + t_i t_j + t_j^2)$. We adopt the shorthand that $P_{i,j} = (t_i + t_j, t_i^2 + t_i t_j + t_j^2)$.

We first classify the collinear triples.

¹Such a set may be constructed greedily; note that given t_1, \dots, t_k there are only countably many values t_{k+1}^* such that there exists a nonzero polynomial P with integer coefficients such that $P(t_1, \dots, t_k, t_{k+1}^*) = 0$. Taking t_{k+1}^* away from this countable set completes the proof.

Claim 2.1. *Let i, j, k, l, m, n be distinct indices. Then*

- $P_{i,j}, P_{j,k}, P_{i,k}$ are collinear
- $P_{i,j}, P_{i,k}, P_{i,\ell}$ are not collinear
- $P_{i,j}, P_{j,k}, P_{k,\ell}$ are not collinear
- $P_{i,j}, P_{i,k}, P_{\ell,m}$ are not collinear
- $P_{i,j}, P_{k,\ell}, P_{m,n}$ are not collinear

Proof. Recall that three points $(x_i, y_i) \in \mathbb{R}^2$ are collinear if and only if

$$\det \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} = 0.$$

For the first part of Claim 2.1, we have that

$$\det \begin{pmatrix} 1 & t_i + t_j & t_i^2 + t_i t_j + t_j^2 \\ 1 & t_i + t_k & t_i^2 + t_i t_k + t_k^2 \\ 1 & t_j + t_k & t_j^2 + t_j t_k + t_k^2 \end{pmatrix} = \det \begin{pmatrix} 1 & t_i + t_j & t_i^2 + t_i t_j + t_j^2 \\ 0 & t_k - t_j & (t_k - t_j)(t_i + t_j + t_k) \\ 0 & t_k - t_i & (t_k - t_i)(t_i + t_j + t_k) \end{pmatrix} = 0$$

as desired.

For the second claim, taking $P_{i,j} = (x_1, y_1)$, $P_{i,k} = (x_2, y_2)$ and $P_{i,\ell} = (x_3, y_3)$ note that

$$\det \begin{pmatrix} 1 & t_i + t_j & t_i^2 + t_i t_j + t_j^2 \\ 1 & t_i + t_k & t_i^2 + t_i t_k + t_k^2 \\ 1 & t_i + t_\ell & t_i^2 + t_i t_\ell + t_\ell^2 \end{pmatrix} \neq 0.$$

This is seen via examining the coefficient of $t_\ell^2 t_k$ which is nonzero, since the only contribution to the determinant comes from the main diagonal in the Laplace expansion. For the third bullet, $P_{i,j} = (x_1, y_1)$, $P_{j,k} = (x_2, y_2)$ and $P_{k,\ell} = (x_3, y_3)$ we see that coefficient of $t_\ell^2 t_k$ is nonzero via an analogous argument with the main diagonal. For the fourth bullet, $P_{i,j} = (x_1, y_1)$, $P_{i,k} = (x_2, y_2)$ and $P_{\ell,m} = (x_3, y_3)$; we see that coefficient of $t_\ell^2 t_k$ is nonzero analogously. Finally for the fifth bullet with $P_{i,j} = (x_1, y_1)$, $P_{k,\ell} = (x_2, y_2)$ and $P_{m,n} = (x_3, y_3)$ we have that the coefficient of $t_m^2 t_i$ is nonzero analogously. By the algebraic independence of t_i , each of the determinants is nonzero, as desired. \square

Remark. The model proceeded slightly differently here and instead more directly computes the determinant in one case and otherwise exhibits a valuation which is nonzero in the remaining cases.

Given Claim 2.1, we complete our proof by arguing that the constructed A satisfies both properties of Theorem 1.1. For the first property, fix any set of points $P \subseteq A$; this point set can be naturally identified with a graph G having $|P|$ edges. Since any graph G with $|P|$ edges has a bipartite subgraph G' with $\geq |P|/2$ edges, we have the desired first property.

For the second property, suppose that A could be partitioned into m sets, each of which has no three collinear points. By Claim 2.1, this would correspond to an m -coloring of $K_{\mathbb{N}}$ with no monochromatic triangle; this contradicts Ramsey's theorem, and in particular the finiteness of $R(3, \dots, 3)$.

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REFERENCES

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- [3] Christian Reiher, Vojtěch Rödl, and Marcelo Sales. Colouring versus density in integers and Hales–Jewett cubes. *J. Lond. Math. Soc. (2)* 110 (2024) 1–24. DOI: [10.1112/jlms.12987](https://doi.org/10.1112/jlms.12987). [1](#)

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