IMPROVED SHORT GAPS BETWEEN PRIMES

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Abstract. We prove that $\liminf_{n\to\infty}(p_{n+1}-p_n) \le 186$, where p_n denotes the nth prime. The improvement combines the equidistribution estimates of Polymath 8a and Stadlmann with complementary factorization conditions that make certain products, and more generally suitable least common multiples of divisor products, triply densely divisible. This permits a larger support for the multidimensional Selberg sieve and an improved numerical optimization, establishing $\mathrm{DHL}[40,2]$ and hence the stated bound. The proof is due to GPT 6 Astra.

1. Introduction

1.1. **Background and main result.** Write $H_m := \liminf_{n\to\infty}(p_{n+m}-p_n)$, where p_n is the nth prime. The twin prime conjecture asserts that $H_1 = 2$. Zhang [21, Thm. 1] proved the first unconditional bound $H_1 < 70\,000\,000$, building on the sieve method of Goldston, Pintz and Yıldırım [6, Thm. 2], who proved

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = 0. \tag{1.1}$$

Zhang's distribution estimates go beyond the Bombieri–Vinogradov range for a restricted family of moduli. Polymath 8a [15] strengthened these estimates and refined the sieve, reducing the bound to 4680.

The multidimensional Selberg sieve of Maynard [12, Thms. 1.1 and 1.3], developed independently by Tao, gave $H_1 \le 600$ and $H_m < \infty$ for every fixed m, using only the Bombieri–Vinogradov theorem. Further refinements by Polymath 8b [16, Thm. 1.4(i)] gave

$$H_1 \le 246. \tag{1.2}$$

We exploit the stronger equidistribution estimates of Polymath 8a [15] and Stadlmann [19] through sieve weights whose divisor supports make fuller use of the available factorizations of the moduli.

A set of distinct integers is *admissible* if it omits a residue class modulo every prime. Following Polymath [16, Claim 3.1], write $\mathrm{DHL}[k,j]$ for the assertion that every admissible k-tuple has infinitely many translates containing at least j primes.

Theorem 1.1. *The assertion* $\mathrm{DHL}[40,2]$ *holds.*

The 40-element set

$$\begin{aligned}\mathcal{H} := \{&0,2,6,12,20,26,30,32,36,42,48,50,56,60,\\ &68,72,78,86,90,92,98,102,110,116,120,126,132,138,\\ &140,146,152,156,158,162,168,170,176,180,182,186\}\end{aligned} \tag{1.3}$$

is admissible and has diameter 186, so Theorem 1.1 implies the following.

Corollary 1.2. *We have* $H_1 \le 186$.

In independent concurrent work, Stadlmann [20] proves $H_1 \le 240$.

We have a Lean 4 formalization of Theorem 1.1 conditional on numerical integral and cap bounds and on finite-field exponential-sum estimates ultimately coming from Deligne's results on the Riemann hypothesis over finite fields [10, 5]. The GitHub repository `openai/PrimeGaps186` contains

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this formalization, `PrimeGaps186.lean`, with the precise exponential-sum bounds of Lemma A.4 stated as explicit assumptions, and the Python-FLINT program `prime_gap_186_certificate.py`, which verifies the numerical bounds. We also provide the accompanying PDF, *Numerical certificate for prime gaps at most 186* [14], which gives the coefficient tables, interval algorithms and complete verification records.

1.2. **Outline of the proof.** Fix an admissible tuple $\mathcal{H} = \{h_1, \dots, h_k\}$. By translation, assume $h_i \geq 0$; admissibility and prime-containing translates are preserved. Write $\mathbf{d} = (d_1, \dots, d_k)$ for the vector of divisors d_j and, for each fixed i, write $\mathbf{e} = (e_j)_{j \neq i}$ for the corresponding vector with its ith coordinate omitted. There is one outer Selberg sum A and, for each coordinate i, two inner sums with nested allowed divisor supports:

$$\begin{aligned} A(n) &\coloneqq \sum_{\substack{d_j \mid n+h_j \\ 1 \leq j \leq k}} \lambda_{\mathbf{d}}, \\ B_i^{(a)}(n) &\coloneqq \sum_{\substack{e_j \mid n+h_j \\ j \neq i}} \beta_{\mathbf{e}}^{(a,i)} \qquad (a = 0, 1), \\ C_i(n) &\coloneqq B_i^{(1)}(n) - B_i^{(0)}(n). \end{aligned} \tag{1.4}$$

The coefficients come from fixed smooth profile functions after finite-band approximation of their dependence on prime fragments (normalized logarithms of the divisor prime factors), and the supported divisor products are squarefree. Both inner sums approximate the same restriction of A obtained when its ith divisor is forced to be 1. The base sum $B_i^{(0)}$ uses a smaller region; $B_i^{(1)}$ retains more of that approximation. The correction C_i is their difference, not a second outer weight.

Put $w_0 \coloneqq \log\log\log x$ and $W \coloneqq \prod_{p \leq w_0} p$. Throughout, x is sufficiently large in terms of the fixed tuple $\mathcal{H}$; in particular, every prime dividing a nonzero difference $h_i - h_j$ divides W. Choose $b_0 \pmod W$ with $(b_0 + h_i, W) = 1$ for every i; admissibility and the Chinese remainder theorem permit this choice. Write $P \coloneqq \mathbf{1}_{\mathbb{P}}$ and let $\sum_n'$ mean summation over $x \leq n < 2x$, $n \equiv b_0 \pmod W$. Our nonnegative weight is always $A(n)^2$. We seek

$$S_2 \coloneqq \sum_{i=1}^{k} \sum_n{}' P(n+h_i) A(n)^2 > S_1 \coloneqq \sum_n{}' A(n)^2, \tag{1.5}$$

which guarantees a translate containing at least two primes. The first approximation uses square completion:

$$A^2 = 2AB_i^{(0)} - (B_i^{(0)})^2 + (A - B_i^{(0)})^2. \tag{1.6}$$

After multiplication by $P(n + h_i)$, the last square is nonnegative. Discarding it leaves mixed and inner products, whose moduli are smaller than those of the full outer square. This is the enlarged-support argument of Polymath 8b [16, §5.3]; we retain this argument and enlarge the allowed coefficient supports. The unrounded divisor-product bounds are:

Sum	A	$B_i^{(0)}$	$B_i^{(1)}$
Product Bound	$x^{.2742997}$	$x^{.2499106033}$	$x^{.251}$

(1.7)

The exact construction takes slightly smaller cutoffs. These bounds concern divisor products, not individual prime factors. They force $d_i = 1$ whenever $n + h_i$ is prime. Ignoring $W = x^{o(1)}$, the outer square has modulus bound $x^{.5485994}$, within the range of ordinary progression counting for S_1 but beyond the prime-distribution range used here. The corresponding bounds for $AB_i^{(0)}$ and $(B_i^{(0)})^2$ are only $x^{.5242103033}$ and $x^{.4998212066}$. The latter is covered by Bombieri–Vinogradov; the former requires the stronger distribution estimates and suitable factorizations of its moduli.

This factorization requirement is where the new support conditions enter. After expansion and the Chinese remainder theorem, a surviving mixed term has modulus

$$WQ, \qquad Q \coloneqq [D, E],$$

where D and E are the outer and inner coefficient-divisor products, and $[\cdot]$ denotes their lcm. A Y-densely divisible integer has divisors within a factor Y of prescribed sizes. Higher orders require the resulting factors to admit further controlled factorizations. We use the singly and doubly densely divisible estimates of Polymath 8a [15, Thm. 2.8], together with the triply densely divisible adaptation of Stadlmann's bilinear estimate indicated in [19, Prop. 1 and §2.1.1]. The details of the latter adaptation are given in §A.4.

The MPZ application in Polymath 8b [16, Thm. 3.5(ii)] uses smooth coefficient divisors. That paper explicitly notes that the stronger dense-divisibility input is not exploited there. It also proposes using that input to enlarge the support, but identifies the difficulty of evaluating the resulting integrals when k is around 50 [16, §9, item 2]. Our conditions instead control the factorizations of the actual combined modulus. Small prime factors from either divisor product can provide the divisors needed to accommodate a larger prime. For example, the prime-factor criterion of Sarajian and Weingartner [17, Thm. 2] gives triple dense divisibility of a squarefree Q when

$$p^3 \le YQ_{<p} \qquad (p \mid Q,\ p > Y), \tag{1.8}$$

where $Q_{<p}$ is the product of its prime factors smaller than p. Proposition 2.3 supplies separate, complementary conditions on D and E which imply this criterion for $[D, E]$, including when D and E share prime factors. Neither integer has to be Y-smooth. This permits additional choices of smooth profile functions without increasing the quoted distribution exponents.

The larger approximation $B_i^{(1)}$ uses a prime minorant. Stadlmann's fixed construction [19, Prop. 2] has exponent of distribution 0.5253 on sufficiently smooth squarefree moduli and retains at least $1 - 2 \cdot 10^{-5}$ of the asymptotic prime mass. It refines Baker and Irving's use of Harman's sieve [2, 7]. The Buchstab-and-discarding procedure has an earlier antecedent in Heath-Brown and Iwaniec [9]; Heath-Brown's identity [8] also enters the distribution estimates. We use the same minorant on the required densely divisible moduli. On the interval in question write it as $\varrho = P - \mathfrak{b}$, with $\mathfrak{b} \ge 0$. Here $\mathfrak{b} = \varrho^- = \max\{-\varrho, 0\}$ is supported on composites, since $\varrho = 1$ on primes and $\varrho \le 0$ on composites. The coefficient primes are kept below the roughness threshold of its support, so $d_i = 1$ there as well.

The product $AB_i^{(1)}$ has modulus bound $x^{.5252997}$, just beyond the endpoint 0.525 of the full-prime estimates used here. The minorant evaluates its mixed correlation, and hence the correlation of $A - B_i^{(0)}$ with C_i. It does not replace P in the positive weight defining S_2. Instead, Cauchy–Schwarz is applied separately with the nonnegative weights P and $\mathfrak{b}$. An upper-bound sieve controls the $\mathfrak{b}$-weighted residual square. All inner pairings have modulus bound $x^{.502}$; a common order-two estimate of level 0.5062 evaluates them for both P and ϱ, and subtraction gives the corresponding $\mathfrak{b}$-weighted terms. These estimates recover a positive part of the residual prime square in (1.6), with a small loss from the minorant's negative part.

§2 gives the divisor support conditions and the distribution estimates. §3 evaluates the sieve weights and bounds the contribution of the minorant's negative part. §4 chooses the weights, verifies their supports and proves the strict integral inequality which gives $S_2 > S_1$ for $k = 40$.

2. Equidistribution conditions and minorants

The mixed sieve sums produce moduli $W[D, E]$, where D and E are squarefree divisor products. We first give separate conditions on D and E which ensure dense divisibility of $[D, E]$. We then state the equidistribution estimates and the properties of the prime minorant. The exponents and positive margins in the analytic estimates are fixed as $x \to \infty$. Implied constants may depend on these parameters and on the fixed coefficient bounds.

2.1. **Divisor support conditions.** Dense divisibility. Write $P^+(m)$ for the largest prime factor of m, with $P^+(1) = 1$. An integer is Y-smooth if $P^+(m) \le Y$. Smoothness allows one to choose divisors close to any prescribed size; the following weaker condition retains precisely this flexibility.

Definition 2.1 ([15, Def. 2.1])**.** For $Y \ge 1$, put $\mathcal{D}^{(0)}(Y) \coloneqq \mathbb{Z}_{>0}$. For an integer $r \ge 1$, a positive integer m belongs to $\mathcal{D}^{(r)}(Y)$ if, for every pair of integers $j, k \ge 0$ with $j + k = r - 1$ and every $1 \le U \le Ym$, there is a factorization

$$m = uv, \qquad U/Y \le v \le U, \qquad u \in \mathcal{D}^{(j)}(Y), \quad v \in \mathcal{D}^{(k)}(Y).$$

We call these integers r-tuply Y-densely divisible, or singly, doubly and triply densely divisible when $r = 1, 2, 3$, respectively.

The definition requires every allocation of the remaining orders $j + k = r - 1$. This is the starred class $\mathcal{D}^*_{r,Y}$ of Sarajian and Weingartner [17, §1.1]. Every Y-smooth integer belongs to all these classes.

We use the elementary properties

$$\mathcal{D}^{(r+1)}(Y) \subseteq \mathcal{D}^{(r)}(Y), \qquad d \mid m,\ m \in \mathcal{D}^{(r)}(Y) \implies d \in \mathcal{D}^{(r)}(Ym/d),$$

and the analogous implication $m \in \mathcal{D}^{(r)}(Y)$, $m \mid n \implies n \in \mathcal{D}^{(r)}(Yn/m)$. Also, the least common multiple of two singly Y-densely divisible integers is singly Y-densely divisible. These facts are contained in [15, Lem. 2.10(0)–(iii)]. In particular, dense divisibility need not pass to a divisor with the same value of Y; the factor m/d must be retained.

For positive squarefree D, put $D_{<p} \coloneqq \prod_{q \mid D,\ q<p} q$. The following prime-factor test will be convenient.

Lemma 2.2. *Let $r \ge 1$ be an integer, $Y \ge 2$, and let Q be a positive squarefree integer. If*

$$p^r \le YQ_{<p} \qquad (p \mid Q,\ p > Y),$$

then $Q \in \mathcal{D}^{(r)}(Y)$. Moreover, if $Q \in \mathcal{D}^{(r)}(Y)$ and W is squarefree, Y-smooth and coprime to Q, then $WQ \in \mathcal{D}^{(r)}(Y)$.

Proof. The first assertion is the lower inclusion in [17, Thm. 2]. For the second, let $N \in \mathcal{D}^{(r)}(Y)$ and $p \le Y$, $p \nmid N$. Induct on r, with order zero immediate. Given $j + k = r - 1$ and $1 \le U \le YNp$, factor N at target U when $U \le YN$ and adjoin p to the first factor; when $U \ge p$, use target U/p and adjoin p to the second. Induction preserves the required orders, and the ranges overlap because $p \le YN$. Iterating proves the assertion about W. At order one, the same two-range argument permits any $p \nmid N$ with $p \le YN$. □

Lemma 2.2 reduces triple dense divisibility to an inequality for each prime factor. We obtain this inequality by splitting p^3 into two factors, one controlled by D and the other by E. For any prime p, write $D_{\ge p} \coloneqq D/D_{<p}$ for the product of the prime factors of D which are at least p, and use the same notation for E and Q. The following proposition allows the split of p^3 to vary with p.

Proposition 2.3. *Let $Y \ge 2$ and $X, A, B > 0$, with $AB \le XY$. Let $f, g : [1, \infty) \to [1, \infty)$ be nondecreasing functions satisfying $f(t)g(t) = t^3$. Suppose the positive squarefree integers D, E satisfy*

$$\begin{aligned} f(p)D_{\ge p} &\le A, \quad g(p) \le B \qquad (p \mid D,\ p > Y), \\ g(p)E_{\ge p} &\le B, \quad f(p) \le A \qquad (p \mid E,\ p > Y). \end{aligned} \tag{2.1}$$

If $Q \coloneqq [D, E] > X$, then $Q \in \mathcal{D}^{(3)}(Y)$. Moreover, if $d \mid D$ and $e \mid E$, then (2.1) *holds with d, e in place of D, E, with the same functions and constants.*

Proof. Fix a prime $p \mid Q$ with $p > Y$, and first suppose $p \mid D$. The first condition gives $f(p)D_{\geq p} \leq A$. If E has a prime factor at least p, let q be the least one. Then $E_{\geq p} = E_{\geq q}$ and $g(p)E_{\geq p} \leq g(q)E_{\geq q} \leq B$. If it has none, $E_{\geq p} = 1$ and the bound $g(p) \leq B$ applies. Multiplying the two bounds gives

$$p^3 D_{\geq p} E_{\geq p} \leq AB \leq XY.$$

Since $Q_{\geq p}$ divides $D_{\geq p}E_{\geq p}$ and $Q > X$,

$$p^3 \leq \frac{XY}{Q_{\geq p}} = \frac{X}{Q} Y Q_{<p} < Y Q_{<p}.$$

The same argument with D, E interchanged handles the remaining primes. Lemma 2.2 proves the assertion. Deleting prime factors only decreases $D_{\geq p}$ and $E_{\geq p}$, which proves the last claim. $\square$

The bounds $g(p) \leq B$ and $f(p) \leq A$ cover the case where the other integer has no prime factor left at that scale. No coprimality assumption on D, E is needed. The freedom to choose f and g, rather than fixing one power of p for each, is used in the numerical optimization.

For single and double dense divisibility, bounds involving only $pD_{\geq p}$ and $pE_{\geq p}$ suffice.

Proposition 2.4. *Let $Y \geq 2$, $X > 0$ and $D_0, E_0 \geq 1$. Suppose $D \leq D_0$ and $E \leq E_0$ are positive and squarefree, $Q := [D, E] > X$, and*

$$pD_{\geq p} \leq \frac{XY}{E_0} \qquad (p \mid D,\ p > Y). \tag{2.2}$$

(i) If $E_0^2 \leq XY$, then $Q \in \mathcal{D}^{(1)}(Y)$.

(ii) If

$$pE_{\geq p} \leq \frac{XY}{D_0} \quad (p \mid E,\ p > Y), \qquad D_0^2 \leq XYE_0, \quad E_0^2 \leq XYD_0,$$

then $Q \in \mathcal{D}^{(2)}(Y)$.

Proof. Since $D > X/E_0$, condition (2.2) gives $p < YD_{<p}$ for every $p \mid D$ with $p > Y$. Thus, D and each of its initial prime products are singly dense. In (i), every prime of E is at most $E_0 \leq XY/E_0 < YD$, so order-one insertion adjoins the missing primes.

For (ii), use the following order-two insertion step, stated in [17, Prop. 1]. If N is doubly Y-dense, $d \mid N$ is singly Y-dense, and $p > P^+(N)$ satisfies $p \leq Y \min\{d, N/d\}$, then Np is doubly Y-dense. Indeed, a singly dense divisor in $[U/Y, U]$ is supplied by N when $U \leq YN$. When $U \geq p^2$, take a singly dense $v \mid N$ at target U/p; then $Yv \geq U/p \geq p$, so order-one insertion makes pv singly dense. In the remaining interval $YN < U < p^2$, the singly dense divisor pd works, since $U/Y < p^2/Y \leq pd \leq YN < U$. Applying the same construction at target YNp/U and taking the complement gives the other recursive orientation.

Now insert the distinct primes of Q increasingly, starting with its Y-smooth part. Just before inserting $p \mid D$, $p > Y$, set $N := Q_{<p}$ and $d := D_{<p}$. By induction N is doubly dense, while $d \mid N$ is singly dense and $p < Yd$. If E has a prime factor at least p, the least such prime gives $pE_{\geq p} \leq XY/D_0$, exactly as in the preceding proof. Since $Q_{\geq p} \leq D_{\geq p}E_{\geq p}$,

$$p \leq \frac{XY}{D_0 E_{\geq p}} < \frac{YQ}{DE_{\geq p}} \leq Y\frac{N}{d}.$$

Otherwise $N/d = Q/D > X/D_0$, while

$$p \leq \sqrt{XY/E_0} \leq XY/D_0 < Y\frac{N}{d}.$$

The first inequality uses $D_{\geq p} \geq p$ and the second uses $D_0^2 \leq XYE_0$. The insertion step applies; interchanging D, E handles the remaining primes. $\square$

If $d \mid D$ and $e \mid E$, the prime-factor inequalities above remain true with d, e in place of D, E, with X, Y, D_0, E_0 unchanged. The conclusion applies whenever $[d, e] > X$. There is also a simple exception for small divisors. If $D \le D_0$ and $E \le E_0$, then $D \le X/E_0$ or $E \le X/D_0$ implies $[D, E] \le X$. Thus, when imposing either proposition only above X, we may admit these small divisors without further restrictions. Intersecting the resulting conditions for the different modulus ranges still gives supports closed under taking divisors.

2.2. **Equidistribution estimates.** For finitely supported sequences α, β on $\mathbb{N}$, define their Dirichlet convolution by

$$(\alpha * \beta)(n) := \sum_{uv=n} \alpha(u)\beta(v).$$

For $(a, q) = 1$, put

$$\Delta(f; a \bmod q) := \sum_{n \equiv a\ (q)} f(n) - \frac{1}{\varphi(q)} \sum_{(n,q)=1} f(n), \tag{2.3}$$

where φ is the Euler totient function. Write Λ for the von Mangoldt function and P for the prime indicator, and set $\Lambda_x := \Lambda \mathbf{1}_{[x,2x)}$ and $P_x := P\mathbf{1}_{[x,2x)}$.

Write $\tau(n) := \sum_{d|n} 1$ for the divisor function. The sequences below may depend on x; all constants in their defining bounds are uniform within the family under consideration. We now recall some definitions from the surrounding literature:

Definition 2.5. A family of sequences $\alpha : \mathbb{N} \to \mathbb{C}$ is *divisor-bounded* if, for some fixed $C \ge 0$,

$$|\alpha(n)| \ll \tau(n)^C$$

uniformly in n and x.

Definition 2.6 ([15, Def. 2.5])**.** A *coefficient sequence* is a finitely supported sequence $\alpha : \mathbb{N} \to \mathbb{C}$ such that

$$|\alpha(n)| \ll \tau(n)^{C_1} (\log x)^{C_2}$$

for fixed $C_1, C_2 \ge 0$, uniformly in n and x.

The cited real-valued estimates apply to complex sequences by separating real and imaginary parts.

Definition 2.7 ([15, Def. 2.5(i)])**.** A sequence α is *located at scale* $N = N(x) \ge 1$ if $\operatorname{supp} \alpha \subseteq [cN, CN]$ for fixed $0 < c < C < \infty$. We also say that α is *at scale* N.

Definition 2.8 ([15, Def. 2.5(iii)]; [19, Def. 2])**.** A sequence α is *smooth at scale* $N = N(x) \ge 1$ if $\alpha(n) = \psi_x(n/N)$, where $\psi_x \in C^\infty(\mathbb{R})$ is supported in a fixed interval $[c, C] \subset (0, \infty)$ and, for every fixed integer $j \ge 0$,

$$\|\psi_x^{(j)}\|_\infty \ll_j (\log x)^{C_j}$$

for a fixed $C_j \ge 0$. The profile ψ_x may depend on x, but c, C, C_j and the implied constants do not.

Definition 2.9 ([15, Def. 2.5(ii)])**.** A coefficient sequence β at scale N has the *Siegel–Walfisz property* if there is a fixed constant $C_{\mathrm{SW}} \ge 0$ such that, for every fixed $L > 0$,

$$\left| \sum_{\substack{n \equiv a\ (q) \\ (n,s)=1}} \beta(n) - \frac{1}{\varphi(q)} \sum_{(n,qs)=1} \beta(n) \right| \ll_L \tau(qs)^{C_{\mathrm{SW}}} N (\log x)^{-L} \tag{2.4}$$

uniformly for all integers $q, s \ge 1$ and all $a \bmod q$ with $(a, q) = 1$. The constant C_{SW} depends only on the coefficient family, not on L, and the implied constant may depend on L and that family but not on x, N, q, s or a.

Proposition 2.10. *Let α, β be coefficient sequences at scales M, N, with $MN \ll x^{C_0}$ for a fixed C_0. Then $\alpha * \beta$ is a coefficient sequence at scale MN. The Siegel–Walfisz property holds uniformly for*
$$\beta(n) = P(n)\mathbf{1}_J(n), \qquad J \subseteq [N, 2N], \qquad N \geq x^{\eta},$$
where J is an interval and $\eta > 0$ is fixed.

Proof. Convolution closure is [15, proof of Lem. 3.4(i)]. The prime-interval assertion is the localization used in [16, §4.5, proof of (63)]; we record its uniformity in the auxiliary coprimality condition.

For the prime-interval assertion, the classical Siegel–Walfisz theorem and partial summation give the estimate uniformly in the two endpoints when q is bounded by a fixed power of $\log x$. For $q > (\log x)^{L+2}$, use the bound $O(N/q + 1 + N/\varphi(q))$ and $q/\varphi(q) \leq \tau(q)$. The condition $(n, s) = 1$ removes at most $\tau(s)$ primes. Since $N \geq x^{\eta}$, both this error and the term 1 are absorbed by (2.4), with $C_{\mathrm{SW}} = 1$. ☐

Let I be a finite set of primes and $P_I := \prod_{p \in I} p$. Write $\mathcal{D}_I^{(r)}(Y) := \{q : q \mid P_I,\ q \in \mathcal{D}^{(r)}(Y)\}$. Fix one primitive class $a \bmod P_I$ and use its restriction for every divisor q of P_I. For $\omega, \delta, \varepsilon > 0$ write
$$\mathcal{E}_r(f; \omega, \delta, \varepsilon) := \sum_{\substack{q \leq x^{1/2+2\omega-\varepsilon} \\ q \mid P_I,\ q \in \mathcal{D}^{(r)}(x^{\delta})}} |\Delta(f; a \bmod q)|. \tag{2.5}$$
The set I and the class a may depend on x; the estimates are uniform in them. The residue class is fixed *outside* the modulus sum. We place a maximum inside that sum only when it is explicitly present, as in Bombieri–Vinogradov below. The fixed retreat ε leaves room for the small factors arising from W and from subdivision of the coefficients. Whenever the parameter inequalities are strict, we may first increase ω and δ slightly. The resulting fixed margin also permits a cutoff $x^{1/2+2\omega}L_0(x)$ for any prescribed positive $L_0(x)$ with $\log L_0(x) = o(\log x)$; constants may depend on that choice.

Proposition 2.11. *Let f be a coefficient sequence at scale x, let $0 < \theta < 1$, and let $\mathcal{Q} \subseteq [1, x^{\theta}] \cap \mathbb{N}$. Choose a primitive residue class $a_q \bmod q$ for each $q \in \mathcal{Q}$. If, for every fixed $L > 0$,*
$$\sum_{q \in \mathcal{Q}} |\Delta(f; a_q \bmod q)| \ll_L x(\log x)^{-L},$$
then, for every fixed $B, L > 0$,
$$\sum_{q \in \mathcal{Q}} \tau(q)^B |\Delta(f; a_q \bmod q)| \ll_{B,L} x(\log x)^{-L}.$$
The conclusion retains the uniformity of the hypothesis. The implication also holds for maxima over the same prescribed sets of primitive classes.

Proof. Put $E(q) := |\Delta(f; a_q \bmod q)|$. Shiu's estimate [18, Thm. 1, (2.1)–(2.2)], applied to a fixed power of the divisor function on an interval of length comparable to x, gives $E(q) \ll x\varphi(q)^{-1}(\log x)^{O(1)}$ uniformly for $q \leq x^{\theta}$. Together with the divisor moments in [15, Lem. 1.3(ii)], this yields
$$\sum_{q \in \mathcal{Q}} \tau(q)^{2B} E(q) \ll x(\log x)^{O_B(1)}.$$
For the mean term use $q/\varphi(q) \leq \tau(q)$. Cauchy–Schwarz now gives
$$\left(\sum_q \tau(q)^B E(q)\right)^2 \leq \left(\sum_q E(q)\right)\left(\sum_q \tau(q)^{2B} E(q)\right).$$
Choose a sufficiently strong unweighted logarithmic saving. The progression bound is uniform in the primitive class, so the same proof applies to the stated maxima. ☐

Proposition 2.12. *Let $0<\theta<1$, let $\mathcal{Q}\subseteq[1,x^\theta]\cap\mathbb{N}$, and choose primitive classes $a_q \bmod q$. Let w be a nonnegative divisor-bounded sequence.*

(i) Suppose f is a coefficient sequence and, for every fixed $L>0$,

$$\sup_{x\le t\le 2x}\sum_{q\in\mathcal{Q}} w(q)|\Delta(f\mathbf{1}_{[x,t)};a_q \bmod q)| \ll_L x(\log x)^{-L}. \tag{2.6}$$

If $\psi\in C^1([x,2x])$ satisfies $\|\psi\|_\infty+\int_x^{2x}|\psi'(t)|\,\mathrm{d}t\ll(\log x)^C$ for a fixed C, then, for every fixed $L>0$,

$$\sup_{x\le u\le v\le 2x}\sum_{q\in\mathcal{Q}} w(q)|\Delta(f\psi\mathbf{1}_{[u,v)};a_q \bmod q)| \ll_L x(\log x)^{-L}.$$

(ii) For a fixed coefficient sequence $f=f_x$, changing each endpoint of an interval in $[x-O(1),2x+O(1)]$ by $O(1)$ changes the corresponding weighted discrepancy sum by at most $x^{\theta+o(1)}$.

(iii) If $\theta<2/3$ and (2.6) holds for $f=\Lambda_x$, then it also holds for $f=P_x$.

These assertions also hold with maxima over prescribed primitive classes when the hypotheses use the same maxima.

Proof. We use partial summation and the divisor bounds in [15, Lem. 1.3]. For (i), put $F_q(t)=\Delta(f\mathbf{1}_{[x,t)};a_q \bmod q)$. Partial summation gives

$$\Delta(f\psi\mathbf{1}_{[u,v)};a_q \bmod q)=\psi(v)F_q(v)-\psi(u)F_q(u)-\int_u^v F_q(t)\psi'(t)\,\mathrm{d}t.$$

Sum absolute values with weights $w(q)$ and use (2.6) with a sufficiently large L. For (ii), the difference is supported on $O(1)$ integers, each with coefficient $x^{o(1)}$; the divisor bound for $w(q)$ gives the stated error on summing over q.

For (iii), first remove prime powers from Λ_x. A primitive class modulo q has at most $2\tau(q)$ square roots, so squares contribute at most $x^{1/2+o(1)}+x^{\theta+o(1)}$ to the summed discrepancies. There are $O(x^{1/3}\log x)$ prime powers of exponent at least three; their contribution is at most $x^{\theta+1/3+o(1)}$. The mean terms satisfy the same bounds by the divisor moments. These are power savings when $\theta<2/3$, uniformly in the interval. Apply (i) with $\psi(t)=1/\log t$ to the remaining sequence $P(t)\log t$. □

The following proposition records the subdivision argument used in [15, §3, proof of Lem. 2.7, (3.9)–(3.13)] and [16, §4.5, proof of (63)]. We state it for regions defined by a fixed number of monomial inequalities.

Proposition 2.13. *Fix an integer $\ell\ge 2$, $\eta>0$, and a finite list of nonzero vectors $\mathbf{v}_j\in\mathbb{R}^\ell$. Let $\gamma_i:\mathbb{N}\to[0,1]$ $(1\le i\le\ell)$, and let $\Omega_x\subseteq[x^\eta,\infty)^\ell$ be cut out by*

$$x\le t_1\cdots t_\ell<2x,\qquad \prod_{i=1}^{\ell} t_i^{v_{ji}}\ \square_j\ z_j(x)\quad \textit{for every } j,$$

where $z_j(x)>0$ and $\square_j\in\{<,\le,>,\ge\}$. For any region S, write

$$F_S(n):=\sum_{\substack{n_1\cdots n_\ell=n\\(n_1,\,\dots,\,n_\ell)\in S}}\prod_{i=1}^{\ell}\gamma_i(n_i).$$

Let $\mathcal{Q}\subseteq\{1,\,\dots,\,\lfloor x\rfloor\}$ and choose a nonempty set $\mathcal{A}_q$ of primitive classes modulo q for every $q\in\mathcal{Q}$. Write $\mathcal{E}(f):=\sum_{q\in\mathcal{Q}}\max_{a\in\mathcal{A}_q}|\Delta(f;a \bmod q)|$.

Suppose that, for every fixed $A,C>0$, with $h=(\log x)^{-C}$,

$$\mathcal{E}(F_B)\ll_{A,C} x(\log x)^{-A} \tag{2.7}$$

uniformly over $B = \prod_{i=1}^{\ell} [N_i, (1+h)N_i)$ *with* $N_i = x^\eta (1+h)^{m_i}$, $m_i \in \mathbb{Z}_{\geq 0}$ *and* $x/2 \leq N_1 \cdots N_\ell \leq 2x$. *Then* $\mathcal{E}(F_{\Omega_x}) \ll_L x(\log x)^{-L}$ *for every fixed* $L > 0$. *The data* $\gamma_i, z_j, \mathcal{Q}, \mathcal{A}_q$ *may depend on* x*; the conclusion retains the uniformity of* (2.7) *in these data.*

Proof. The proof is essentially identical to the box decomposition in [16, §4.5, proof of (63)]. We allow finitely many monomial boundaries.

Fix $C > 0$ and take x sufficiently large that $(1+h)^\ell \leq 2$. There are $O((h^{-1} \log x)^\ell)$ boxes meeting Ω_x. Let F_0 be the sum of F_B over boxes wholly contained in Ω_x, and let G be the sum of F_B over boxes meeting Ω_x but not wholly contained in it. Since the grid is disjoint,

$$|F_{\Omega_x} - F_0| \leq G.$$

Every box under consideration has $x/2 \leq N_1 \cdots N_\ell \leq 2x$.

Include the two product cutoffs among the boundary inequalities. A boundary box lies within $O(h)$, in logarithmic coordinates, of one of the corresponding hyperplanes. More precisely, for some fixed nonzero vector $\mathbf{v}$ and threshold $z(x)$, every point of the box satisfies

$$\left| \sum_i v_i \log t_i - \log z(x) \right| \leq h \sum_i |v_i|.$$

All integer tuples in these boxes also satisfy $n_i \geq x^\eta$ and $x/2 \leq \prod_i n_i \leq 4x$. Choose i with $v_i \neq 0$ and fix the other $\ell - 1$ variables, whose product we denote by m. The displayed strip confines n_i to an interval of relative width $O(h)$ and with upper endpoint at most $4x/m$ after intersection with the product range. It therefore contains $O(hx/m + 1)$ integers. Necessarily $m \leq 4x^{1-\eta}$. Summing $O(hx/m + 1)$ over the remaining variables, for each of the finitely many boundary hyperplanes, gives by the usual harmonic-sum bounds

$$\sum_n G(n) \ll (hx + x^{1-\eta})(\log x)^{\ell - 1}. \tag{2.8}$$

The constants in the box count and this bound depend on the fixed geometry and ℓ, η, but not on C. If $|R| \leq G$, then

$$|\Delta(R; a \bmod q)| \leq |\Delta(G; a \bmod q)| + \frac{2}{\varphi(q)} \sum_n G(n).$$

Apply this with $R = F_{\Omega_x} - F_0$. The triangle inequality, (2.7), (2.8), and $\sum_{q \leq x} 1/\varphi(q) \ll \log x$ give

$$\mathcal{E}(F_{\Omega_x}) \ll_{A,C} x(\log x)^{\ell(C+1) - A} + (hx + x^{1-\eta})(\log x)^\ell.$$

Choose $C > L + \ell + 2$ and then $A > L + \ell(C+1) + 2$. □

Proposition 2.14. *Let* M *be squarefree. For every* $p \mid M$*, choose a nonempty set* $\mathcal{A}_p \subseteq (\mathbb{Z}/p\mathbb{Z})^\times$*, and put*

$$\mathcal{A} = \{a \bmod M : a \bmod p \in \mathcal{A}_p \text{ for every } p \mid M\}, \qquad m(q) = \prod_{p \mid q} |\mathcal{A}_p| \quad (q \mid M).$$

For any finitely supported f*, any* $\mathcal{Q} \subseteq \{q : q \mid M\}$*, and any nonnegative weights* $w(q)$*,*

$$\sum_{q \in \mathcal{Q}} w(q) \max_{a \in \mathcal{A}} |\Delta(f; a \bmod q)| \leq \frac{1}{|\mathcal{A}|} \sum_{a \in \mathcal{A}} \sum_{q \in \mathcal{Q}} w(q) m(q) |\Delta(f; a \bmod q)|. \tag{2.9}$$

If $|\mathcal{A}_p| \leq K$ *for a fixed* $K \geq 1$*, then* $m(q) \leq \tau(q)^{\log_2 K}$*. Consequently, for coefficient sequences at scale* x *and moduli at most* x^θ, $0 < \theta < 1$*, estimates*

$$\sum_{q \in \mathcal{Q}} |\Delta(f; a \bmod q)| \ll_L x(\log x)^{-L}$$

for every fixed $L > 0$*, uniformly in* $a \in \mathcal{A}$*, imply the same bounds for the displayed maximum with any fixed divisor weights.*

Proof. The proof is essentially identical to the averaging argument in [16, §4.3], with the sets $\mathcal{A}_p$ in place of the residue classes supplied by a particular sieve.

The Chinese remainder theorem gives $m(q)$ images of $\mathcal{A}$ modulo q, each with $|\mathcal{A}|/m(q)$ preimages. The maximum is at most the sum over these images. Multiplying by $w(q)$ and summing gives (2.9). The bound for $m(q)$ follows from $|\mathcal{A}_p| \le K$ and squarefreeness. The last assertion follows from Proposition 2.11 and averaging over $\mathcal{A}$. □

We now state the analytic distribution estimates. Numbered references use the versions specified in the bibliography.

Proposition 2.15. *For every fixed $L, \varepsilon > 0$,*

$$\sum_{q \le x^{1/2-\varepsilon}} \max_{(a,q)=1} |\Delta(\Lambda_x; a \bmod q)| \ll_{L,\varepsilon} x(\log x)^{-L}. \tag{2.10}$$

This estimate is uniform when $[x, 2x)$ is replaced by any of its subintervals. If α, β are coefficient sequences at scales M, N, $MN \asymp x$, $M, N \ge x^{\varepsilon_0}$ for some fixed $\varepsilon_0 > 0$, and β has the Siegel–Walfisz property, then the same estimate holds with $\alpha * \beta$ in place of Λ_x, with the implied constant also depending on ε_0 and the coefficient bounds.

The prime estimate is the usual Bombieri–Vinogradov theorem [4, (1.4)], applied at interval endpoints. The bilinear assertion is [15, Thm. 2.9], whose Definition 2.5(ii) uses the absolute Siegel–Walfisz normalization (2.4). Their logarithmic cutoffs exceed $x^{1/2-\varepsilon}$ for sufficiently large x.

Proposition 2.16 ([15, Thm. 2.8(i),(ii),(iv)])**.** *Suppose $0 < \omega < 1/12$, $0 < \delta < 1/4 + \omega$, $0 < \sigma < 1/2$, and $68\omega + 14\delta < 1$. Let α, β be coefficient sequences at scales M, N, where*

$$MN \asymp x, \qquad x^{1/2-\sigma} \le N \le x^{1/2},$$

and suppose that β has the Siegel–Walfisz property. For $r = 1$ or 2, assume respectively that

$$54\omega + 15\delta + 5\sigma < 1 \quad (r = 1), \qquad 56\omega + 16\delta + 4\sigma < 1 \quad (r = 2). \tag{2.11}$$

Then, for every fixed $L, \varepsilon > 0$,

$$\mathcal{E}_r(\alpha * \beta; \omega, \delta, \varepsilon) \ll x(\log x)^{-L}.$$

This combines [15, Thm. 2.8(i),(ii)] with its Type II estimate, [15, Thm. 2.8(iv)]. The latter holds already on singly densely divisible moduli, and hence also on the doubly densely divisible subclass.

Proposition 2.17 ([15, Thm. 2.8(v)])**.** *Suppose $0 < \omega < 1/12$, $0 < \delta < 1/4 + \omega$ and*

$$\frac{1}{18} + \frac{28}{9}\omega + \frac{2}{9}\delta < \sigma < \frac{1}{2}. \tag{2.12}$$

Let α be a coefficient sequence at scale M, and let ψ_1, ψ_2, ψ_3 be smooth sequences at scales N_1, N_2, N_3, with

$$\begin{gathered} MN_1N_2N_3 \asymp x, \qquad N_iN_j \ge x^{1/2+\sigma} \quad (i \ne j), \\ x^{2\sigma} \le N_i \le x^{1/2-\sigma} \quad (1 \le i \le 3). \end{gathered} \tag{2.13}$$

Then, for every fixed $L, \varepsilon > 0$,

$$\mathcal{E}_1(\alpha * \psi_1 * \psi_2 * \psi_3; \omega, \delta, \varepsilon) \ll x(\log x)^{-L}.$$

Proposition 2.18. *Let $\omega, \delta, \sigma > 0$ satisfy*

$$72\omega + 24\delta < 1, \qquad 48\omega + 16\delta + 4\sigma < 1, \qquad 64\omega + 20\delta + 2\sigma < 1. \tag{2.14}$$

If α, β are coefficient sequences at scales M, N, $MN \asymp x$, $x^{1/2-\sigma} \le N \le x^{1/2}$, and β has the Siegel–Walfisz property, then

$$\mathcal{E}_3(\alpha * \beta; \omega, \delta, \varepsilon) \ll x(\log x)^{-L}$$

for every fixed $L, \varepsilon > 0$.

Proof. The smooth-modulus estimate is [19, Prop. 1]. The extension to triple dense divisibility is proved in §A.4: the divisor choices retain the density loss after division, and the completion retains the original unit restrictions. The resulting common-factor bounds give the unchanged inequalities (2.14). ☐

Corollary 2.19. *Let $\omega, \delta > 0$ and $r \in \{1, 2, 3\}$. Assume the corresponding condition*

$$\begin{aligned} 108\omega + 30\delta &< 1 \quad (r = 1), \\ 280\omega + 80\delta &< 3 \quad (r = 2), \\ 240\omega + 80\delta &< 3 \quad (r = 3). \end{aligned} \tag{2.15}$$

Then, for every fixed $L, \varepsilon > 0$,

$$\mathcal{E}_r(P_x; \omega, \delta, \varepsilon) \ll x(\log x)^{-L}. \tag{2.16}$$

The estimate is uniform on subintervals of $[x, 2x)$, and also holds with Λ_x in place of P_x.

Proof. In each range, $\sigma > 1/10$ may be chosen sufficiently close to $1/10$ that $\sigma > 2\omega$, (2.12), and the required bilinear inequalities all hold. At order three one may take $\sigma := 1/10 + (3 - 240\omega - 80\delta)/40$. For $r = 1, 2$ use Proposition 2.16; for $r = 3$ use Proposition 2.18. Combine these with Proposition 2.17 and the Heath–Brown reduction [15, Lem. 2.7], whose combinatorial input [15, Lem. 3.1] requires $\sigma > 1/10$. The subinterval uniformity follows from [15, Rem. 3.6]; Proposition 2.12(iii) gives the P_x estimate. ☐

For the minorant we also need a smooth-coefficient estimate which reaches shorter factor lengths.

Lemma 2.20. *Let α be a coefficient sequence at scale M and β a smooth sequence at scale N, with $MN \asymp x$ and $x^{1/2-\sigma} \le N \le x^{1/2}$. For $\omega, \delta, \sigma > 0$ satisfying*

$$4\sigma + 28\omega + 8\delta < 1, \tag{2.17}$$

one has $\mathcal{E}_1(\alpha * \beta; \omega, \delta, \varepsilon) \ll x(\log x)^{-L}$ for every fixed $L, \varepsilon > 0$. Without the preceding restriction on N, if $0 < \theta < 1$, $\varepsilon_0 > 0$ and $N \ge x^{\theta+\varepsilon_0}$, then

$$\sum_{q \le x^\theta} \max_{(a,q)=1} |\Delta(\alpha * \beta; a \bmod q)| \ll_{L,\theta,\varepsilon_0} x(\log x)^{-L}. \tag{2.18}$$

Proof. The first assertion is proved in §A.5, adapting [2, Lem. 3] to singly densely divisible moduli.[1] The reciprocal-phase estimate uses the least common multiple of the complete moduli rq_0q_1 and rq_0q_2, which is singly dense by [15, Lem. 2.10(ii)]. For (2.18), the Type 0 argument [15, §3, proof of Lem. 2.7, (3.14) and following] applies because N/q exceeds a fixed power of x. ☐

2.3. **The prime minorant and its negative part.** We use the arithmetic function constructed in [19, Prop. 2 and §4.8], which refines the Harman-sieve minorant of Baker and Irving [2]. For $m \ge 1$, let $\psi(m, z)$ be the indicator that every prime factor of m is at least z, with $\psi(1, z) = 1$. Set

$$\begin{gathered} a_* := 0.40481, \qquad b_* := 0.59519, \qquad c_* := 0.33856, \\ \xi_* := 0.19038, \qquad u_* := 0.23848. \end{gathered} \tag{2.19}$$

These are exact values; in particular $b_* = 1 - a_*$, $\xi_* = 1 - 2a_*$ and $u_* = 1 - 4\xi_*$. In the sums below, all p_j are prime, $\alpha_j := \log p_j / \log x$, and m_5 is a positive integer. For $x \le n < 2x$, put

$$\mathfrak{b}_1(n; x) := \sum_{\substack{n = p_1p_2p_3p_4m_5 \\ \xi_* \le \alpha_4 < \alpha_3 < \alpha_2 < \alpha_1 < a_* \\ \alpha_1 + \alpha_2 < a_* \\ \alpha_2 + \alpha_3 + \alpha_4 > b_*}} \psi(m_5, p_4), \tag{2.20}$$

[1] In the notation of [2, Lem. 3, proof following (2.4)], $(c_1, \delta_1') = (h, rq_1)$, so the ensuing mean term contains (h, rq_1q_2). For $m = rq_1q_2$, the bound $\sum_{1 \le h \le H} (h, m) \le H\tau(m)$ preserves the estimate; see §A.5.

$$\mathfrak{b}_2(n;x) := \sum_{\substack{n=p_2p_3p_4p_5p_6 \\ \xi_*\le\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6\le u_* \\ \alpha_2,\alpha_4>\alpha_3 \\ \alpha_2+\alpha_4<a_* \\ \alpha_2+\alpha_3+\alpha_5>b_* \\ \alpha_6\ge\alpha_5}} 1, \tag{2.21}$$

$$\mathfrak{b}(n;x) := \mathfrak{b}_1(n;x) + \mathfrak{b}_2(n;x), \qquad \varrho(n;x) := P(n) - \mathfrak{b}(n;x). \tag{2.22}$$

We extend these functions by zero outside $[x, 2x)$. The sums count configurations and can have multiplicity. The expression for $\mathfrak{b}_2$ is the one in Stadlmann [19, Prop. 2]; the equivalent form in [19, Lem. 22] interchanges p_2 and p_4.

Proposition 2.21. *For sufficiently large x, the function ϱ equals one on primes in $[x, 2x)$ and is nonpositive on composites. It is supported on integers all of whose prime factors are at least x^{ξ_*}, and*

$$\varrho_- := \max\{-\varrho, 0\} = \mathfrak{b} \ge 0.$$

Every integer in the support of $\mathfrak{b}$ has exactly five prime factors, counted with multiplicity, and $\mathfrak{b}(n;x) \ll 1$. There is a constant $0 \le \kappa_{\mathrm{true}} \le 1/50000$ such that, with $m_{\mathrm{true}} := 1 - \kappa_{\mathrm{true}}$,

$$\begin{aligned} \sum_{x\le n<2x} \varrho(n;x) &= (m_{\mathrm{true}} + o(1))\frac{x}{\log x}, \\ \sum_{x\le n<2x} \mathfrak{b}(n;x) &= (\kappa_{\mathrm{true}} + o(1))\frac{x}{\log x}. \end{aligned} \tag{2.23}$$

Proof. The lower bound for the mass is [19, Prop. 2]. The exceptional sums are nonnegative, vanish on primes, and are supported on x^{ξ_*}-rough integers. In (2.20), the possibility $m_5 = 1$ would give $n < x^{2a_*} < x$. If m_5 were composite, n would have at least six prime factors of size at least x^{ξ_*}, which is impossible for large x since $6\xi_* > 1$. Thus, m_5 is prime, and both exceptional sums have five prime factors. Their multiplicities are bounded by a fixed number of assignments of those factors. The prime number theorem in the two fixed regions of prime variables gives constants $\kappa_1, \kappa_2 \ge 0$ for their asymptotic masses. Set $\kappa_{\mathrm{true}} = \kappa_1 + \kappa_2$ and use Stadlmann's bound $\kappa_{\mathrm{true}} \le 1/50000$ [19, Prop. 2]. □

We next record the distribution range for this *same* arithmetic function on densely divisible moduli. No change to its defining regions or to its mass is made.

Proposition 2.22. *Put $\sigma_m := 1/2 - a_*$. Let $0 < \omega < 1/12$ and $0 < \delta < 1/4 + \omega$ satisfy*

$$\begin{gathered} \frac{1}{4} + 7\omega + 2\delta < c_*, \qquad \frac{1}{18} + \frac{28}{9}\omega + \frac{2}{9}\delta < \frac{19}{200}, \\ \frac{1}{2} + 2\omega < b_*. \end{gathered} \tag{2.24}$$

For $r = 1, 2$, assume $68\omega + 14\delta < 1$ and the corresponding inequality in (2.11) with $\sigma = \sigma_m$. For $r = 3$, assume (2.14) with $\sigma = \sigma_m$. Then, for every fixed $L, \varepsilon > 0$,

$$\mathcal{E}_r(\varrho; \omega, \delta, \varepsilon) \ll x(\log x)^{-L}. \tag{2.25}$$

In particular, the order-three estimate permits

$$\omega = \frac{253}{20000}, \qquad 0 < \delta < 10^{-6}, \qquad \frac{1}{2} + 2\omega = 0.5253.$$

For ordinary moduli one also has

$$\sum_{q\le x^{1/2-\varepsilon}} \max_{(a,q)=1} |\Delta(\varrho; a \bmod q)| \ll_{L,\varepsilon} x(\log x)^{-L}. \tag{2.26}$$

Proof. Choose a fixed $0 < \tau \le 10^{-10}$ sufficiently small that the strict parameter inequalities remain valid with $\sigma_m + \tau$ in place of σ_m and with the three upper bounds in (2.24) decreased by τ. The coefficient construction in Lemma A.8 and corrected Buchstab decomposition in Lemma A.9 prove the version of [19, Lems. 18–22 and §4.8] needed here.[2]

Both proofs are given in §A.6; the defining regions of ϱ are unchanged.

Lemma 2.20 handles smooth factors of length at least $x^{c_*-\tau}$ and at most $x^{1/2}$ by the first inequality of (2.24). Proposition 2.16 or Proposition 2.18, with $\sigma = \sigma_m + \tau$, handles the bilinear range from $x^{a_*-\tau}$ to $x^{b_*+\tau}$. The localized coefficients are those in [19, Lems. 19–20], with the corrected sixth region.[3] They satisfy Siegel–Walfisz uniformly in their endpoints and auxiliary coprimality conditions by Lemma A.8, so the two factors may be interchanged.[4] Smooth factors longer than $x^{b_*+\tau}$ are covered by (2.18), since $1/2 + 2\omega < b_*$.

For the three smooth factors in the Type III terms, choose σ_3 strictly between the second lower bound in (2.24) and 19/200, and decrease τ further so that $\tau < 19/200 - \sigma_3$. Their lengths lie between $x^{0.19-\tau}$ and $x^{0.405+\tau}$, and their pairwise products are at least $x^{0.595-\tau}$. These satisfy (2.13), so Proposition 2.17 applies. These are the three distribution inputs required by the decomposition. Its logarithmically many subdivisions are absorbed by the arbitrary logarithmic saving. Substitution verifies the order-three example; the tolerance is chosen after ω, δ.

For (2.26), apply Proposition 2.13 to $\mathfrak{b}_1$ and $\mathfrak{b}_2$ with $\ell = 5$, $\eta = \xi_*$ and $\gamma_i = P$. Both regions are defined by monomial inequalities, and m_5 in $\mathfrak{b}_1$ is prime by Proposition 2.21. For the box estimate, write $F_B = \alpha * \beta$, grouping four prime factors into α and the fifth into β. Proposition 2.10 makes α a coefficient sequence and gives β the Siegel–Walfisz property. Since their scales satisfy $M \gg x^{4\xi_*}$ and $N \gg x^{\xi_*}$, Proposition 2.15 supplies the uniform box estimate with a maximum over all primitive classes; together with Proposition 2.12(iii), it also handles the prime term. $\square$

For example, the two estimates have the common order-two parameters

$$\omega_0 = \frac{1}{500}, \qquad \delta_0 = \frac{1}{50}, \qquad \frac{1}{2} + 2\omega_0 = \frac{63}{125} = 0.504.$$

The full-prime condition in (2.15) holds at order two, and the minorant conditions hold with $\sigma = \sigma_m$. Thus, both estimates hold on the same class of doubly x^{δ_0}-densely divisible squarefree moduli up to $x^{0.504-\varepsilon}$, with the same coherent residue classes. Subtracting gives the corresponding estimate for $\mathfrak{b} = P_x - \varrho$, whose mass is governed by the small constant κ_{true} in (2.23). The common source used by the numerical trial has the parameters specified in §4.2.3; its support containment is verified there.

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[2] In [19, proof of Lem. 22], use $\sum \alpha_i = \log_x n$. The complementary-cutoff difference has $p_3p_4 \in [x^{a_*}, 2x^{a_*})$ and $p_1p_2 \in (x^{b_*}, 2x^{b_*})$, and is covered by the widened Type II estimate. On the fixed-cutoff region $n_1 = n/(p_2p_3p_4) > x^{c_*}$, the correct reversal is

$$P(n_1) = \psi(n_1, x^{\xi_*}) - \sum_{\substack{n_1 = p_5p_6 \\ x^{\xi_*} \le p_5 \le p_6}} 1,$$

where p_5, p_6 are primes, possibly equal: $6\xi_* > 1$ excludes longer rough composite cofactors. Without an indexed subset sum in $[a_*, b_*]$, every pair sum is below a_*, giving $\alpha_i < a_* - \xi_* < u_*$. Lemma A.9 justifies the stated exceptional sum and repeated-prime diagonal, with errors summed over moduli. These corrections retain the same minorant.

[3] In [19, Lem. 19(6)], replace $\xi_* \le \alpha_4 < \alpha_3 < \alpha_2 < a_*$ by $\xi_* \le \alpha_3 < \alpha_2 < a_*$, retaining $\alpha_4 \ge \alpha_3$ and all other restrictions. No order between α_2 and α_4 is imposed. This is the region used in the proof of Lem. 22; the grouping $I = \{3, 4\}$, $J = \{2\}$ in the proof of Lem. 19 remains valid. The coefficient verification is in Lemma A.8.

[4] The nonsmooth factor in [2, Lem. 5], invoked by [19, Lem. 18(I)], must satisfy Siegel–Walfisz. The identity $\mu_{<z} = \mu * \psi(\cdot, z)$, where $\mu_{<z}(n) := \mu(n)\mathbf{1}_{P^+(n)<z}$ and $z = x^{\xi_*}$, does not by itself remove the first-crossing and prime-order restrictions. Lemma A.8 proves the property for the actual localized coefficients, uniformly in endpoints and auxiliary coprimality, with one fixed divisor exponent, and controls the associated boundary majorants. The corresponding hypothesis is explicit in [19, proof of Lem. 20, property (3)].

To control the negative part after multiplication by sieve weights, its small unweighted mean alone is not enough. We use the following pointwise majorant, whose proof is elementary.

Proposition 2.23. *For squarefree $n \in [x, 2x)$, define*

$$N_2(n) := \#\{p < q : \ pq \mid n, \ p, q \geq x^{\xi_*}, \ pq < x^{a_*}\}.$$

For sufficiently large x,

$$0 \leq \varrho_-(n; x) = \mathfrak{b}(n; x) \leq \frac{12}{5} N_2(n). \tag{2.27}$$

There are $O(x^{1-\xi_*})$ nonsquarefree integers in the support of $\mathfrak{b}$ in $[x, 2x)$. Their contribution still has a fixed power saving after multiplication by fixed divisor-bounded weights.

Proof. It suffices to consider $\mathfrak{b}(n; x) > 0$. Rank its five distinct prime factors increasingly, and let E be the set of rank pairs whose product is less than x^{a_*}. This set is downward closed and $|E| = N_2(n)$. The ordering restrictions in (2.20)–(2.21) give

$$\mathfrak{b}_1 \leq 2\mathbf{1}_{(4,5)\in E} + \mathbf{1}_{(3,5)\in E} + \mathbf{1}_{(3,4)\in E}, \qquad \mathfrak{b}_2 \leq 2 \sum_{(r,s)\in E} (r-1).$$

For $\mathfrak{b}_1$ there are four possible omitted factors. For a marked pair (r, s) in $\mathfrak{b}_2$ there are two orientations, $r-1$ choices of the smaller third factor, and a prescribed order for the remaining pair. If $(4, 5) \in E$, then $|E| = 10$ and these bounds sum to 24. If $(3, 5) \in E$ but $(4, 5) \notin E$, then $|E| = 9$ and they sum to 16. If $(3, 4) \in E$ but neither preceding case occurs, then $|E| \geq 6$ and they sum to at most 11. Otherwise $\mathfrak{b}_1 = 0$ and the bound for $\mathfrak{b}_2$ is at most $2|E|$. This proves (2.27). A nonsquarefree exception is divisible by p^2 for some $p \geq x^{\xi_*}$; summing $2x/p^2$ over such primes gives the asserted count. □

3. Evaluation of the sieve weights

We evaluate the divisor sums for general coefficient weights, following the diagonal change of variables in [12, §5]. The distribution estimates enter through the moduli produced by pairs of coefficients. We then express the main terms as integrals and prove the Selberg upper bound needed for the negative part of the minorant. The choice of supports and the final combination are left to §4.

3.1. Coefficients and diagonal variables.

Fix an admissible tuple $\mathcal{H} = \{h_1, \ldots, h_k\}$, with $k \geq 2$. Retain W, b_0, w_0 and $\sum_n'$ from the introduction. By our largeness convention, every prime divisor of a nonzero $h_i - h_j$ divides W. Fix $\rho_*, \zeta > 0$ with $\rho_*\zeta < 1$, and set

$$\begin{aligned} R &= x^{\rho_*}, \qquad & B_x &= \frac{\varphi(W)}{W}\log x, \qquad & B_R &= \rho_* B_x, \\ C_x &= \frac{x}{W} B_x^{-k}, \qquad & \mathcal{C}_x &= \frac{x}{W} B_R^{-k} = \rho_*^{-k} C_x, \\ \mathcal{P}_x &= \frac{x}{\varphi(W)\log x} B_R^{-(k-1)} = \rho_*^{-(k-1)} C_x = \rho_* \mathcal{C}_x. \end{aligned} \tag{3.1}$$

All parameters and support bounds in this section are fixed as $x \to \infty$.

For $1 \leq m \leq k$, let $\mathcal{R}_m$ be the set of tuples $\mathbf{r} = (r_1, \ldots, r_m)$ for which $r_1 \cdots r_m$ is squarefree, coprime to W, and has no prime factor above R^ζ. Arrays indexed by $\mathcal{R}_m$ are extended by zero to all of $\mathbb{N}^m$.

Definition 3.1. A diagonal array $y = (y_\mathbf{r})$ is a real array, possibly depending on x, with a fixed bound $\|y\|_\infty \leq M$ and

$$y_\mathbf{r} = 0 \qquad \text{if } r_1 \cdots r_m > R^S$$

for some fixed $S > 0$. Its coefficient weights are

$$\lambda_{\mathbf{d}}^{(m)}(y) = B_R^{-m} \left(\prod_{j=1}^{m} \mu(d_j) d_j \right) \sum_{\substack{\mathbf{r} \in \mathcal{R}_m \\ d_j \mid r_j \ (1 \le j \le m)}} \frac{y_{\mathbf{r}}}{\prod_{j=1}^{m} \varphi(r_j)}. \tag{3.2}$$

For two arrays of the same dimension define

$$\langle y, z \rangle_{R,m} = \frac{1}{B_R^m} \sum_{\mathbf{r} \in \mathcal{R}_m} \frac{y_{\mathbf{r}} z_{\mathbf{r}}}{\prod_{j=1}^{m} \varphi(r_j)}, \qquad \|y\|_{R,m}^2 = \langle y, y \rangle_{R,m}. \tag{3.3}$$

Lemma 3.2. *The weights in* (3.2) *satisfy*

$$y_{\mathbf{r}} = B_R^m \left(\prod_{j=1}^{m} \mu(r_j) \varphi(r_j) \right) \sum_{\substack{\mathbf{d} \\ r_j \mid d_j \ (1 \le j \le m)}} \frac{\lambda_{\mathbf{d}}^{(m)}(y)}{\prod_{j=1}^{m} d_j}. \tag{3.4}$$

If y is supported in a set $\mathcal{S}$ such that $\mathbf{r} \in \mathcal{S}$ and $d_j \mid r_j$ for every j imply $\mathbf{d} \in \mathcal{S}$, then $\lambda^{(m)}(y)$ is supported in the same set. Moreover,

$$|\lambda_{\mathbf{d}}^{(m)}(y)| \ll_{m,\zeta} \|y\|_\infty \frac{D}{\varphi(D)}, \quad D = \prod_j d_j, \qquad \|y\|_{R,m} \ll_{m,\zeta} \|y\|_\infty.$$

Proof. The inversion is the change of variables in [12, §5, (5.7)–(5.9)], with our factor B_R^{-m}. A nonzero coefficient requires a tuple $\mathbf{r}$ with $y_{\mathbf{r}} \neq 0$ and $d_j \mid r_j$ for every j, which proves the support assertion. Write $M_R = \prod_{p \le R^\zeta,\, p \nmid W} (1 + 1/(p-1))$. Putting $r_j = d_j t_j$ bounds the coefficient by

$$B_R^{-m} \|y\|_\infty \frac{D}{\varphi(D)} M_R^m \ll \|y\|_\infty \frac{D}{\varphi(D)}.$$

For the norm bound, drop the coprimality conditions between coordinates and use Mertens' formula

$$\frac{1}{B_R} \prod_{\substack{p \le R^\zeta \\ p \nmid W}} \left(1 + \frac{1}{p-1} \right) \longrightarrow e^\gamma \zeta, \tag{3.5}$$

where γ is the Euler–Mascheroni constant. □

Fix $1 \le i \le k$. For a k-coordinate array y and a $(k-1)$-coordinate array z, define

$$\begin{aligned} A_y(n) &= \sum_{d_j \mid n + h_j \ (1 \le j \le k)} \lambda_{\mathbf{d}}^{(k)}(y), \\ B_{z,i}(n) &= \sum_{d_j \mid n + h_j \ (j \neq i)} \lambda_{\mathbf{d}_{\hat{i}}}^{(k-1)}(z). \end{aligned} \tag{3.6}$$

Here the coordinates of z are indexed, in their usual order, by $\{1, \ldots, k\} \setminus \{i\}$. These sums are $x^{o(1)}$ pointwise for $n \asymp x$, by Lemma 3.2 and the divisor bound. For $\mathbf{r}_{\hat{i}} \in \mathcal{R}_{k-1}$ put

$$(y^{(i)})_{\mathbf{r}_{\hat{i}}} = \frac{1}{B_R} \sum_{a \ge 1} \frac{y_{r_1, \ldots, r_{i-1}, a, r_{i+1}, \ldots, r_k}}{\varphi(a)}. \tag{3.7}$$

The zero extension of y enforces squarefreeness, the prime cap, and $(a, W \prod_{j \neq i} r_j) = 1$ in this sum.

Lemma 3.3. *The array $y^{(i)}$ has bounded supremum and the same total support bound as y. If $A_{0i,y}$ denotes the sum A_y with $d_i = 1$, then*

$$A_{0i,y} = B_{y^{(i)},i}. \tag{3.8}$$

Consequently, $A_y(n) = B_{y^{(i)},i}(n)$ whenever $n + h_i$ has no prime factor at most R^ζ.

Proof. When $d_i = 1$, the definition of the coefficients gives

$$\lambda_{\mathbf{d}}^{(k)}(y) = \lambda_{\mathbf{d}_{\hat{i}}}^{(k-1)}(y^{(i)}).$$

Indeed, the sum over the ith diagonal index is $B_R y^{(i)}$, which cancels one factor B_R^{-1} in the coefficient normalization. The other divisibility conditions are unchanged. Summing over $d_j \mid n + h_j$, $j \neq i$, proves (3.8). The supremum bound follows from (3.5). If $n + h_i$ has no prime factor at most R^ζ, the prime-factor restriction on the coefficients forces $d_i = 1$. □

3.2. **Counting and prime-weighted sums.**

Proposition 3.4 ([12, Lem. 5.1, (5.13)])**.** *Let y be a diagonal array on $\mathcal{R}_k$, supported on $r_1 \cdots r_k \leq R^S$. If $2\rho_* S < 1$, then*

$$\sum_n{}' A_y(n)^2 = \mathcal{C}_x(\|y\|_{R,k}^2 + o(1)). \tag{3.9}$$

The error is uniform for fixed supremum and support bounds.

The cited algebra applies before any smooth profile is chosen. Use the coefficient radius R^S and the factor B_R^{-k} in (3.2). The ordinary counting error is $O(R^{2S}(\log x)^{O(1)})$; the strict inequality $2\rho_* S < 1$ makes it smaller than any fixed logarithmic multiple of $\mathcal{C}_x$.

For two arrays y, z on the coordinates other than i, define $\mathcal{Q}_i(y,z)$ to consist of the moduli

$$q = W \prod_{j \neq i} [d_j, e_j] = W[D,E], \qquad D = \prod_{j \neq i} d_j, \quad E = \prod_{j \neq i} e_j, \tag{3.10}$$

where $\lambda_{\mathbf{d}}^{(k-1)}(y)\lambda_{\mathbf{e}}^{(k-1)}(z) \neq 0$ and the $[d_j, e_j]$ are pairwise coprime. Let $\mathcal{I}$ contain the primes at most R^ζ and the primes dividing W.

Proposition 3.5. *Let y, z be diagonal arrays on $\mathcal{R}_{k-1}$. Let c_x be the restriction to $[x, 2x)$ of $P = \mathbf{1}_{\mathbb{P}}$ or of $\varrho(\cdot; x)$, with the same interval used in its discrepancy. Put $\mu_c = 1$ in the first case and $\mu_c = 1 - \kappa_{\mathrm{true}}$ in the second, where κ_{true} is the actual deficit in Proposition 2.21. For the minorant assume also $\rho_* \zeta < \xi_*$. Suppose $\mathcal{Q}_i(y,z) \subseteq [1, x^\theta]$ for fixed $0 < \theta < 1$ and, for every fixed $L > 0$,*

$$\sup_{(a, P_{\mathcal{I}})=1} \sum_{q \in \mathcal{Q}_i(y,z)} |\Delta(c_x; a \bmod q)| \ll_L x(\log x)^{-L}. \tag{3.11}$$

Then

$$\sum_n{}' c_x(n + h_i) B_{y,i}(n) B_{z,i}(n) = \mathcal{P}_x(\mu_c \langle y, z \rangle_{R,k-1} + o(1)). \tag{3.12}$$

If u is a diagonal array on $\mathcal{R}_k$ and the same hypotheses hold with $y = u^{(i)}$, then

$$\sum_n{}' c_x(n + h_i) A_u(n) B_{z,i}(n) = \mathcal{P}_x(\mu_c \langle u^{(i)}, z \rangle_{R,k-1} + o(1)). \tag{3.13}$$

The modulus family is independent of the global residue class. Each error is uniform for fixed array bounds and uniform constants in (3.11)*. Fixed endpoint shifts do not change the result.*

Proof. We use the diagonal calculation of [12, Lems. 5.2–5.3] and the residue-class averaging of [16, §4.3], with (3.11) as the distribution input. We record the bilinear calculation for signed arrays depending on individual prime factors.

The support of c_x is coprime to every modulus in (3.10): for primes this follows from $\rho_* \zeta < 1$, and for the minorant from roughness and $\rho_* \zeta < \xi_*$. Thus, a compatible coefficient pair contributes its common mean

$$\frac{\sum_m c_x(m)}{\varphi(W) \prod_{j \neq i} \varphi([d_j, e_j])}$$

and a discrepancy in a primitive residue class. Factor this common mean out before summing the signed coefficients. The fixed endpoint shifts are negligible by Proposition 2.12(ii).

For a prime $p \mid q/W$, the detected variable $n+h_i$ lies in one of the $k-1$ distinct nonzero classes $h_i - h_j \pmod p$. Let $\mathcal{A}_i$ be the primitive classes modulo $P_{\mathcal{I}}$ satisfying

$$a \equiv b_0 + h_i \pmod W,$$
$$a \bmod p \in \{h_i - h_j : j \neq i\} \qquad (p \in \mathcal{I},\ p \nmid W).$$

Every class from a compatible coefficient pair extends to $\mathcal{A}_i$. At each prime there are $k-1$ coordinate choices and three membership patterns, left, right, or both roots. If $t = \omega(q/W)$, Proposition 2.14 replaces the maximum over compatible classes by an average over $\mathcal{A}_i$, with factor $(k-1)^t$. The total loss is at most $[3(k-1)^2]^t$, a fixed divisor power. The family being residue-independent, Lemma 3.2, Proposition 2.11, and (3.11) give an arbitrary logarithmic saving for the summed discrepancy. It remains to evaluate the coefficient sum

$$M(y,z) = \sum_{\mathbf{d},\mathbf{e}}{}^{*} \frac{\lambda_{\mathbf{d}}^{(k-1)}(y)\lambda_{\mathbf{e}}^{(k-1)}(z)}{\prod_{j\neq i}\varphi([d_j,e_j])},$$

where the asterisk requires the $[d_j,e_j]$ to be pairwise coprime. We claim that

$$M(y,z) = B_R^{-2(k-1)} \sum_{\mathbf{r}\in\mathcal{R}_{k-1}} \frac{y_{\mathbf{r}} z_{\mathbf{r}}}{\prod_{j\neq i}\varphi(r_j)} + O_{k,\zeta}\left(B_R^{-(k-1)}\|y\|_\infty \|z\|_\infty \sum_{p>w_0} \frac{1}{p^2}\right). \tag{3.14}$$

This is the bilinear diagonal calculation underlying [12, (5.21)–(5.26) and Lem. 5.3]. The following prime-by-prime calculation also explains why it applies to arrays depending on the individual prime factors.

For $p > w_0$ and $j \neq i$, put

$$U_{p,j}(a) = \frac{1 - p\mathbf{1}_{a\equiv -h_j \pmod p}}{p-1}.$$

After factoring out $B_R^{-2(k-1)}$, inversion expresses $M(y,z)$ as a sum of products of these functions, averaged over $a \not\equiv -h_i \pmod p$ at each prime. Put $a_p = p-1$. For this average $\mathbb{E}_{p,i}$,

$$\mathbb{E}_{p,i}U_{p,j} = -a_p^{-2}, \qquad \mathbb{E}_{p,i}U_{p,j}^2 = a_p^{-1} + a_p^{-2} - a_p^{-3},$$
$$\mathbb{E}_{p,i}(U_{p,j}U_{p,l}) = -a_p^{-2} - 2a_p^{-3} \qquad (j \neq l).$$

Each diagonal tuple contains a prime in at most one coordinate, so only these first and second moments occur in the bilinear product. Replace these averages by ones in which the events indexed by j are independent and each has probability $1/p$. Then the first moments vanish and the second moments are $\mathbf{1}_{j=l}/(p-1)$, giving the displayed diagonal sum. The sum of the absolute errors at one prime is $O_k(p^{-2})$. Expanding the difference of the two products therefore bounds the total error by

$$\ll_k B_R^{-2(k-1)}\|y\|_\infty\|z\|_\infty \prod_{w_0<p\le R^\zeta}\left(1 + \frac{k-1}{p-1} + O_k(p^{-2})\right)\sum_{p>w_0} p^{-2}.$$

Mertens' formula bounds the product by $O_{k,\zeta}(B_R^{k-1})$, proving (3.14). Since $\sum_{p>w_0} p^{-2} \to 0$, the main term in the sieve sum is

$$\frac{\mu_c x}{\varphi(W)\log x} B_R^{-(k-1)}(\langle y,z\rangle_{R,k-1} + o(1)),$$

which proves (3.12). Finally, apply Lemma 3.3 for (3.13). Only the mixed moduli are used for that assertion; no estimate for the outer self-square is required. □

Condition (3.11) follows from the distribution results of §2 when the actual moduli lie in a fixed finite union of the permitted residue-independent families, with fixed positive retreats. Proposition 2.12 supplies the prime-indicator and endpoint transfers. The modulus to check is $W[D,E]$, including the factor W.

3.3. **Smooth profiles and their quadratic forms.** We now choose the diagonal arrays by sampling fixed smooth functions. The calculation is analogous to [12, Lems. 6.1–6.3]: harmonic sums over the diagonal indices become integrals, and summing out one index becomes a partial integral. We also record the larger prime factors of each index, since the dense-divisibility conditions depend on those factors.

Fix $0 < u_0 < \zeta$. A divisor coordinate is $X = (t_0; u_1, \ldots, u_m)$, where $t_0 \geq 0$ and the fragments $u_j \in (u_0, \zeta]$ are unordered. Write $|X| = t_0 + \sum_j u_j$. For a squarefree integer r with $P^+(r) \leq R^\zeta$, define

$$X_R(r) = \left(\sum_{\substack{p|r \\ p \leq R^{u_0}}} \log_R p;\ (\log_R p)_{p|r,\ p>R^{u_0}} \right).$$

Thus, t_0 records the total size of the smaller prime factors, the u_j record the remaining prime factors individually, and $|X_R(r)| = \log_R r$.

Let ρ_{D} be the continuous Dickman function, determined by $\rho_{\mathrm{D}}(t) = 1$ on $[0,1]$ and $t\rho_{\mathrm{D}}'(t) = -\rho_{\mathrm{D}}(t-1)$ for $t > 1$. The measure used below is the sum, over all $m \geq 0$, of

$$\mathrm{d}\nu(X) = \rho_{\mathrm{D}}(t_0/u_0)\,\mathrm{d}t_0\,\frac{1}{m!}\prod_{j=1}^{m}\frac{\mathrm{d}u_j}{u_j}. \tag{3.15}$$

The factor $1/m!$ accounts for the unordered fragments; the empty product is one. This is a finite measure, not a probability measure. It has mass $e^\gamma\zeta$, where γ is Euler's constant, and its image under $X \mapsto |X|$ is

$$\rho_{\mathrm{D}}(t/\zeta)\,\mathrm{d}t. \tag{3.16}$$

Reducing u_0 and then adding the newly recorded fragments back into the small total gives the original measure.

Definition 3.6. A smooth profile in d coordinates is a real function $F(X_1, \ldots, X_d)$ with the following properties. For every fixed choice of fragment counts $m_1, \ldots, m_d$, it is the restriction of a function in $C_c^\infty(\mathbb{R}^{d+\sum_j m_j})$, symmetric in the fragments within each coordinate. Also, for some fixed S, it vanishes when $\sum_j |X_j| > S$; we call this a total support bound of S. Only count tuples with $\sum_j m_j \leq S/u_0$ can contribute. No compatibility between different count tuples is required. Its diagonal array is

$$y[F]_{\mathbf{r}} = F(X_R(r_1), \ldots, X_R(r_d)) \qquad (\mathbf{r} \in \mathcal{R}_d),$$

extended by zero elsewhere. When $d = k$, write $y^{(i)}[F] = (y[F])^{(i)}$ for the discrete marginal in (3.7). In the corresponding dimensions, put $A_F = A_{y[F]}$, $A^0_{F,i} = A_{0i,y[F]}$ and $B_{G,i} = B_{y[G],i}$. The parameters and profiles are fixed as $x \to \infty$.

For a profile F in k coordinates, put

$$I(F) = \int F^2\,\mathrm{d}\nu^k, \tag{3.17}$$

$$F_i(\mathbf{X}_{\hat{i}}) = (E_iF)(\mathbf{X}_{\hat{i}}) = \int F(X_1, \ldots, X_{i-1}, X, X_{i+1}, \ldots, X_k)\,\mathrm{d}\nu(X), \tag{3.18}$$

$$J^{(i)}(F) = \int F_i^2\,\mathrm{d}\nu^{k-1}. \tag{3.19}$$

We also write $V_i = F_i$ and $J_{\mathrm{full},i} = J^{(i)}$. Unsubscripted norms and inner products use the appropriate product of ν. If F depends only on the totals $|X_j|$ and $S \leq \zeta$, then (3.16) is Lebesgue measure throughout its support. The forms are then the usual Maynard integrals, after rescaling the total support to one. The individual fragments allow the additional prime-factor restrictions needed here.

Proposition 3.7. *Fix $1 \le i \le k$. Let F be a smooth profile in k coordinates and G, H smooth profiles in $k-1$ coordinates, using the same u_0, ζ. Their sampled arrays have uniformly bounded supremum and retain their total support bounds. The function F_i is a smooth profile, with the same total support bound as F, and*

$$\begin{aligned} \|y[F]\|_{R,k}^2 &\longrightarrow I(F), \\ \langle y[G], y[H]\rangle_{R,k-1} &\longrightarrow \langle G, H\rangle, \\ \langle y^{(i)}[F], y[G]\rangle_{R,k-1} &\longrightarrow \langle F_i, G\rangle. \end{aligned}$$

Moreover,

$$\|y^{(i)}[F] - y[F_i]\|_{R,k-1}^2 \longrightarrow 0. \tag{3.20}$$

Consequently, $\|y^{(i)}[F] - y[G]\|_{R,k-1}^2 \to \|F_i - G\|_2^2$. If F has total support at most S, then

$$J^{(i)}(F) \le SI(F), \qquad \sum_{i=1}^{k} J_{\mathrm{full},i}(F) \le kSI(F). \tag{3.21}$$

Proof. We first establish the harmonic summation formula for these coordinates. Decompose each squarefree index uniquely as $r = ap_1 \cdots p_m$, where $P^+(a) \le R^{u_0}$ and $R^{u_0} < p_j \le R^{\zeta}$. The normalized small-cofactor measure has Laplace transform

$$\frac{1}{B_R} \prod_{w_0 < p \le R^{u_0}} \left(1 + \frac{e^{-s \log_R p}}{p-1}\right) \longrightarrow e^{\gamma} u_0 \exp\left(\int_0^{u_0} \frac{e^{-su} - 1}{u} \,\mathrm{d}u\right) \qquad (s \ge 0).$$

Mertens' formulas [13, (5), (13), (15)] supply the value at $s = 0$ and the prime harmonic integral. After division by that value, each Euler factor is $1 + (e^{-s\log_R p} - 1)/p$; expanding its logarithm has error $O(\sum_{p > w_0} p^{-2}) = o(1)$. The Dickman transform [1, (2.4)–(2.6)] identifies the limit as the transform of $\rho_{\mathrm{D}}(t_0/u_0)\,\mathrm{d}t_0$.

Mertens' formula also gives weak convergence of $\sum_{R^{u_0} < p \le R^{\zeta}} (p-1)^{-1} \delta_{\log_R p}$ to $\mathrm{d}u/u$ on $(u_0, \zeta]$, where δ_t is unit mass at t. For each fixed count, allowing repeated large primes costs $O_m(\sum_{p > R^{u_0}} (p-1)^{-2}) = o(1)$ against bounded tests. Divide the ordered sums by $m!$ to obtain (3.15). There are only finitely many contributing count tuples, and all endpoints have zero limiting measure, including $t_0 = 0$. The arithmetic atom $a = 1$ has mass $1/B_R \to 0$. Summing over m inserts the factor $\exp(\int_{u_0}^{\zeta} e^{-su}\,\mathrm{d}u/u)$ into the small-cofactor transform. The resulting total-size transform is $e^{\gamma}\zeta \exp(\int_0^{\zeta} (e^{-su} - 1)\,\mathrm{d}u/u)$, which proves (3.16) and the stated mass. The same calculation between two cutoffs proves the assertion about adding fragments to the small total.

Before imposing coprimality, the normalized harmonic measure is the product of the one-coordinate measures. Imposing coprimality between two coordinates changes a bounded normalized sum by $O(\sum_{p > w_0} p^{-2}) = o(1)$: under the full squarefree harmonic weights, a prime p occurs in each coordinate with probability $1/p$. A union bound over primes and pairs of coordinates therefore gives

$$\frac{1}{B_R^d} \sum_{\mathbf{r} \in \mathcal{R}_d} \frac{F(X_R(r_1), \ldots, X_R(r_d))}{\prod_j \varphi(r_j)} \longrightarrow \int F \,\mathrm{d}\nu^d. \tag{3.22}$$

Apply this formula to the products in the three pairings. Expanding (3.7) gives the mixed-marginal limit; opening its square gives two independently integrated coordinates and proves $\|y^{(i)}[F]\|_{R,k-1}^2 \to \|F_i\|_2^2$. These two coordinates are each coprime to the retained tuple, but are not required to be coprime to one another.

Bounded sampling follows from the finitely many smooth functions and $|X_R(r)| = \log_R r$. The full harmonic mass also gives $\|y^{(i)}[F]\|_\infty \le (e^{\gamma}\zeta + o(1))\|F\|_\infty$. For each choice of retained fragment counts, F_i is a finite sum of integrals over fixed domains. Differentiating compactly supported smooth extensions under those integrals proves smoothness; its support lies in a finite union of compact projections. Taking $G = F_i$ in the preceding limits and expanding the squared difference now proves (3.20). Finally, the permitted mass of the integrated coordinate is at most $\int_0^S \rho_{\mathrm{D}}(t/\zeta)\,\mathrm{d}t \le S$. Cauchy–Schwarz gives $J^{(i)}(F) \le SI(F)$. $\square$

Lemma 3.8 (Bounded profiles and exact faces)**.** *The harmonic limits above also hold for fixed bounded profiles of bounded total support whose discontinuity sets have limiting measure zero, sampled by the same rule. Their marginals are bounded and have null discontinuity sets; smoothness is asserted only for the smooth profiles of Definition 3.6. For fixed finite real linear combinations z_x, z_x' of sampled d-coordinate arrays and exact discrete marginals, with corresponding combinations of profiles and marginals H, H', one has*

$$\langle z_x, z_x' \rangle_{R,d} = \langle H, H' \rangle + o(1). \tag{3.23}$$

In particular, for such an outer profile F of total support S, $2\rho_* S < 1$ gives

$$\sum_n{}' A_F(n)^2 = \mathcal{C}_x(I(F) + o(1)).$$

Under the actual-modulus hypotheses of Proposition 3.5, the mixed pairing with a canonical inner profile H is

$$\sum_n{}' c_x(n + h_i) A_F(n) B_{H,i}(n) = \mathcal{P}_x(\mu_c \langle F_i, H \rangle + o(1)).$$

The coefficient roots retain every hereditary support and prime cap.

Proof. The weak harmonic convergence in (3.22) applies to bounded tests with null discontinuity sets. Fubini and dominated convergence give the same property for each marginal. In an exact-face square, expand the erased sums first: the retained root is shared, whereas the two erased roots may share primes with each other. Only their intersections with retained roots, and intersections between retained coordinates, are forbidden. Removing these restrictions costs $O(\sum_{p > w_0} p^{-2}) = o(1)$ in the normalized bounded form, as in the preceding proof. Bilinearity gives (3.23), without a pointwise limit for the discrete marginal. Lemmas 3.2–3.3 preserve the support bounds; Propositions 3.4–3.5 give the two arithmetic forms. $\square$

For the fragment calculations below, it is convenient to retain all fragments. Let Π_ζ be the Poisson process on $(0, \zeta]$ with intensity $\mathrm{d}u/u$, and put $\nu_\zeta = e^\gamma \zeta\, \mathcal{L}(\Pi_\zeta)$. Its projection to the seed and the fragments above u_0 is precisely (3.15); profiles are identified with their pullbacks. The standard Poisson and Dickman identities [11, Thm. 3.9] [1, (2.4)–(2.6)] give, for $0 < z \le \zeta$,

$$\nu_\zeta\big|_{\{\max X \le z\}} = \nu_z. \tag{3.24}$$

Thus, an inward prime cap uses ν_z, with no extra z/ζ factor; totals above z retain their Dickman density. Conditionally on total $t \le \zeta$, ranked fragments have law $t\,\mathrm{PD}(1)$; for $t > \zeta$ this law is conditioned on its largest fragment being at most ζ [1, Thm. 3.1, Cors. 3.1–3.2].

For ordered distinct occurrences $\mathbf{u} \in X^r_{\neq}$, let $X \setminus \mathbf{u}$ delete the marked occurrences. The reduced Mecke identity [11, Thm. 4.5] states that, for every nonnegative measurable Ψ,

$$\int \sum_{\mathbf{u} \in X^r_{\neq}} \Psi(\mathbf{u}, X \setminus \mathbf{u})\,\mathrm{d}\nu_\zeta(X) = \int_{(0,\zeta]^r} \int \Psi(\mathbf{u}, X)\,\mathrm{d}\nu_\zeta(X) \prod_{j=1}^r \frac{\mathrm{d}u_j}{u_j}. \tag{3.25}$$

Unordered marking contributes $1/r!$ only for symmetric integrands; labelled roles retain the ordered form.

3.4. **A Selberg upper bound for the negative part.** To count the rough cofactor left after a prime pair is marked, we use the classical one-dimensional Selberg coefficients. Their diagonal variables are constant, so their energy is exact. Recall $B_x = B_R/\rho_*$.

Lemma 3.9. *Fix $z > 0$, put $Z = x^z$, and define*

$$G_Z := \sum_{\substack{a\le Z\\(a,W)=1}} \frac{\mu^2(a)}{\varphi(a)}, \qquad \lambda_d^Z := \frac{\mu(d)d}{G_Z} \sum_{\substack{a\le Z\\ d\mid a\\(a,W)=1}} \frac{\mu^2(a)}{\varphi(a)}, \qquad L_Z(t) := \sum_{d\mid t} \lambda_d^Z.$$

These coefficients vanish unless $d \le Z$ is squarefree and coprime to W, and

$$\lambda_1^Z = 1, \qquad |\lambda_d^Z| \le d/\varphi(d), \qquad G_Z = (z + o(1))B_x.$$

Their ordinary diagonal variables are exactly

$$z_a := \mu(a)\varphi(a) \sum_{a\mid d} \frac{\lambda_d^Z}{d} = \frac{\mu^2(a)\mathbf{1}_{a\le Z,\,(a,W)=1}}{G_Z}.$$

In particular,

$$\sum_{d,e} \frac{\lambda_d^Z\lambda_e^Z}{[d,e]} = \sum_a \frac{z_a^2}{\varphi(a)} = \frac{1}{G_Z}, \qquad \sup_a |z_a| = G_Z^{-1} \ll_z B_x^{-1}.$$

If $z < \xi$, then $L_Z(t) = 1$ on every x^ξ-rough positive integer.

Here, z_a is the ordinary diagonal variable. In the normalization of (3.2), the corresponding raw array is $B_R z_a$, after enlarging the ambient prime cap if necessary.

Proof. The diagonal and energy identities are the classical Selberg identities [12, §5]. Writing $a = db$ bounds the inner sum by $G_Z/\varphi(d)$, and on a rough integer only $d = 1$ contributes.

For the asymptotic, apply [12, Lem. 6.1] with $\gamma(p) = \mathbf{1}_{p\nmid W}$ and test function 1. Its singular product is $\varphi(W)/W$; its constants A_1, A_2 may be fixed, and its parameter L is $O(\log w_0)$, since $W = \prod_{p\le w_0} p$ for all sufficiently large x. Thus, uniformly for our choice of W,

$$G_Z = \frac{\varphi(W)}{W}(\log Z + O(\log w_0)) = (z + o(1))B_x.$$

Changing the strict cutoff in that lemma to $a \le Z$ adds at most $Z^{-1+o(1)}$ and does not affect the estimate. ∎

Proposition 3.10. *Fix $r_c, z, \xi > 0$ with $\xi > \max\{\rho_*\zeta, z\}$. Let y be a bounded diagonal array on $\mathcal{R}_{k-1}$, vanishing when $\prod_{j\ne i} r_j > x^{r_c}$. Let $\mathcal{M}$ be any set of prime pairs $p < q$ with $p, q \ge x^\xi$ and $pq \le x^s$, where*

$$s + 2r_c + 2z < 1.$$

Then,

$$\sum_{(p,q)\in\mathcal{M}} \sum_{\substack{x\le n<2x\\ n\equiv b_0 \pmod W\\ pq\mid n+h_i}} L_Z\left(\frac{n+h_i}{pq}\right)^2 B_{y,i}(n)^2 = \mathcal{P}_x\left(\frac{1}{z}\|y\|_{R,k-1}^2 \sum_{(p,q)\in\mathcal{M}} \frac{1}{pq} + o(1)\right).$$

The error is uniform for fixed supremum and support bounds on y. No distribution estimate for primes or for the minorant is assumed.

Proof. We use the ordinary CRT calculation in [16, Thm. 3.6(i) and §4.1], with the exact auxiliary diagonal from Lemma 3.9. We account for the marked pair and for primes shared by auxiliary and retained indices. The marked primes exceed both coefficient prime caps. Hence, when $pq \mid n+h_i$, one has $L_Z((n+h_i)/(pq)) = L_Z(n+h_i)$. Opening both squares gives $x/(Wpq)$ times their complete CRT mean. The coefficient bound in Lemma 3.9 and Lemma 3.2 bound the total counting error by

$$x^{s+2r_c+2z+o(1)} = o(\mathcal{P}_x).$$

For a prime $p > w_0$ and a coordinate j, put $U_{p,j} = (1 - p\mathbf{1}_{p\mid n+h_j})/(p-1)$. Here the average is over all residues modulo p, so

$$\mathbb{E}_p U_{p,j} = 0, \qquad\qquad \mathbb{E}_p U_{p,j}^2 = \frac{1}{p-1}.$$

Put $d = k-1$. If the divisibility events in distinct coordinates were independent, the CRT mean of $L_Z^2 B_{y,i}^2$ would be exactly

$$B_R^{-2d}\left(\sum_a \frac{z_a^2}{\varphi(a)}\right)\left(\sum_{\mathbf{r}} \frac{y_{\mathbf{r}}^2}{\prod_{j\neq i}\varphi(r_j)}\right) = \frac{B_R^{-d}}{G_Z}\|y\|_{R,d}^2.$$

The local states are $S \in \{\varnothing, \{i\}\} \cup \{\{j\}, \{i,j\} : j \neq i\}$; the state $\{i,j\}$ retains a prime shared by the auxiliary index and coordinate j. With $a_p = p-1$, their exact kernel is

$$K_p(S,T) = \frac{1 - |S\cup T| + a_p|S\cap T|}{a_p^{|S|+|T|}}, \qquad D_p(S,T) = \mathbf{1}_{S=T}a_p^{-|S|}$$

in the actual and independent models, respectively. In particular, all mixed third and fourth moments are included. One has $\sum_{S,T}|K_p - D_p| = O_k(p^{-2})$ and $\sum_{S,T}|D_p| = (1+d/a_p)(1+1/a_p)$. Since $|z_a| \leq G_Z^{-1}$, tensor expansion bounds the total difference by

$$\ll \frac{B_R^{-2d}\|y\|_\infty^2}{G_Z^2} \prod_{w_0<p\leq x^{\max\{\rho_*\zeta,z\}}}\left(1 + \frac{k}{p-1} + O_k(p^{-2})\right)\sum_{p>w_0} p^{-2} = o(B_R^{-d}/B_x).$$

The harmonic sum $\sum_{(p,q)\in\mathcal{M}} 1/(pq)$ is $O_{\xi,s}(1)$. Multiplication by x/W, together with $G_Z \sim zB_x$ and $(x/W)B_R^{-d}/B_x = \mathcal{P}_x$, proves the result. $\square$

For the application, fix the global cap

$$\xi_0 = \frac{19037}{100000} < \xi_* = \frac{9519}{50000} \tag{3.26}$$

and impose, on every outer and inner coefficient root, even in the radial cores,

$$P^+\left(\prod_j d_j\right) \leq x^{\xi_0}. \tag{3.27}$$

This hereditary restriction is preserved by the transform in Lemma 3.2. Since $\mathfrak{b}(n;x) > 0$ only on x^{ξ_*}-rough integers, the cap forces $d_i = 1$ on this support. The exact restriction identity (3.8) therefore gives

$$A_F(n) = A_{F,i}^0(n) \qquad \text{whenever } \mathfrak{b}(n+h_i;x) > 0. \tag{3.28}$$

The same identity holds when $n+h_i$ is prime. Roughness also gives, for every actual coefficient modulus q,

$$\sum_{\substack{x\leq n<2x\\(n,q)=1}} \varrho(n;x) = \sum_{x\leq n<2x} \varrho(n;x). \tag{3.29}$$

Proposition 3.11 (Exceptional square bound)**.** *Let C_i be a finite real linear combination, with coefficients independent of x, of canonical $(k-1)$-coordinate sums and exact faces $A^0_{F,i}$ from Lemma 3.8. The profiles are fixed, bounded, and have limiting-null discontinuity sets. Suppose every summand satisfies* (3.27) *and has coefficient-root physical radius at most*

$$r_c = \frac{11}{40}, \qquad K_{\mathrm{ex}} = \frac{17}{50}. \tag{3.30}$$

Let H_i be the corresponding combination of canonical profiles and marginals $F_i = E_i F$. Then

$$\sum_n{}' \mathfrak{b}(n+h_i;x)C_i(n)^2 \le \mathcal{P}_x\big(K_{\mathrm{ex}}\|H_i\|_2^2 + o(1)\big). \tag{3.31}$$

The norm is the full fragment norm. All profiles, bands and auxiliary cutoffs are fixed before $x \to \infty$; their exact face arrays may depend on x.

Proof. By inversion and Lemma 3.3, $C_i = B_{y_x,i}$ for a uniformly bounded raw array of the same radius and cap. In Proposition 3.10 use the ambient cap $\min\{\zeta, \xi_0/\rho_*\}$: the array vanishes outside it, so its weights and diagonal norm are unchanged. (3.23) gives $\|y_x\|^2_{R,k-1} \to \|H_i\|_2^2$. The coefficient and divisor bounds of Lemma 3.2 make the nonsquarefree exceptions in Proposition 2.23, and fixed endpoint shifts, negligible in this weighted sum.

Divide $[2\xi_*, a_*]$ into 1024 equal bins with right endpoints s_j, and set

$$z_j = \frac{1-2r_c-s_j}{2} - \frac{1}{10000}, \qquad Z_j = x^{z_j}. \tag{3.32}$$

These satisfy $4499/200000 \le z_j < 863/25000 < \xi_0 < \xi_*$ and $s_j + 2r_c + 2z_j = 1 - 1/5000$. Thus, Proposition 3.10 applies in each bin. On the exceptional support, $L_{Z_j} = 1$. The positive majorant of Proposition 2.23 gives

$$\mathfrak{b}(n+h_i;x)C_i(n)^2 \le \frac{12}{5}\sum_{j=1}^{1024} \sum_{\substack{p<q,\ p,q\ge x^{\xi_*},\ pq<x^{a_*}\\ s_{j-1}<\log_x(pq)\le s_j}} \mathbf{1}_{pq\mid n+h_i} L_{Z_j}(n+h_i)^2 C_i(n)^2$$

outside the negligible nonsquarefree set. The strict condition $pq < x^{a_*}$ is inherited from N_2 in Proposition 2.23, including in the last bin $s_{1024} = a_*$. Only ordinary counting is used: the error in bin j is $x^{s_j+2r_c+2z_j+o(1)} = o(\mathcal{P}_x)$.

The prime harmonic measure is $\mathrm{d}u/u$ on fixed positive logarithmic intervals [13, (5), (13)]. Integrating over $\xi_* \le u < s/2$ gives the unordered pair measure

$$\mathrm{d}\mu_2(s) = \frac{1}{s}\log\left(\frac{s-\xi_*}{\xi_*}\right)\mathrm{d}s, \qquad 2\xi_* \le s \le a_*. \tag{3.33}$$

The last arithmetic bin has $pq < x^{a_*}$; its limiting mass remains $\mu_2((s_{1023}, a_*])$, because the endpoint has zero μ_2-mass. Proposition 3.10 gives the constant $(12/5)\sum_j z_j^{-1}\mu_2((s_{j-1}, s_j])$ in (3.31). This sum is a right-endpoint bound for

$$\frac{24}{5}\int_{2\xi_*}^{a_*} \frac{\log((s-\xi_*)/\xi_*)}{s(1-2r_c-s-2/10000)}\,\mathrm{d}s. \tag{3.34}$$

We bound the finite bin sum itself. Put $L_{21}(t) = \sum_{m=1}^{21}(-1)^{m+1}t^m/m$, $h_{\mathrm{ex}} = (a_* - 2\xi_*)/1024$, $c_{\mathrm{ex}} = 1 - 2r_c - 2/10000$ and $t_j = (s_j - 2\xi_*)/\xi_* \in [0,1)$. The alternating-series bound gives $\log(1+t) \le L_{21}(t)$ on this interval. The density in (3.33) is increasing, so the finite bin sum is at most

$$\frac{1}{10^{25}}\sum_{j=1}^{1024}\left\lceil 10^{25}\frac{24h_{\mathrm{ex}}L_{21}(t_j)}{5s_j(c_{\mathrm{ex}}-s_j)}\right\rceil = \frac{8403340100682264191104\,01}{25000000000000000000000000} < \frac{17}{50}. \tag{3.35}$$

The exact auxiliary energy in Lemma 3.9 requires no smoothing loss. This proves (3.31). $\square$

4. Choice of weights and completion of the proof

4.1. **The trial function and its supports.** Take the rational midpoint step function F of [14, §§1.1–1.2], with its outer, base, enlarged and full cap domains $\mathscr{H}_O, \mathscr{H}_0, \mathscr{H}_1, \mathscr{H}_{\mathrm{f}}$. The parameters are

$$\begin{aligned} k &:= 40, & S &:= \frac{2742997}{2624989}, & \rho_* &:= \frac{2624989}{10000000}, \\ T_0 &:= \frac{2499106033}{2624989000}, & T_1 &:= \frac{2510000}{2624989}, & \rho &:= \frac{262499}{1000000}. \end{aligned} \tag{4.1}$$

Thus $0 < T_0 < T_1 < S$, and the physical radii satisfy

$$r_O := \rho_* S = \frac{2742997}{10^7} < \frac{11}{40}, \qquad \rho_* T_0 = \frac{2499106033}{10^{10}}, \qquad \rho_* T_1 = \frac{251}{1000} < \frac{11}{40}. \tag{4.2}$$

Write

$$h := S/98304 \tag{4.3}$$

for the trial mesh. The cap domains use the full upper endpoint of each cell, with fragment caps rounded down to this mesh; their exact cells are listed in [14, §1.1]. Their caps decrease with the radial cell index, so the domains are divisor-closed; also $\mathscr{H}_0 \subseteq \mathscr{H}_1 \subseteq \mathscr{H}_{\mathrm{f}}$. All face forms use the same outer trial F.

Use $\nu = \nu_\zeta$ from §3, with $\zeta := \xi_0/\rho_*$. The companion uses the smaller ambient cap ξ_0/ρ, but every retained fragment cap is at most $68225h < \xi_0/\rho$. By (3.24), the restricted measures, and hence the trial forms, coincide without an additional normalization factor.

4.2. **Verification of the supports.**

4.2.1. *Divisor-closed source conditions.* Fix $R > 1$, $\xi \geq 0$ and $Y := R^\xi$. For a positive squarefree integer D, put

$$s_D := \log_R D, \qquad u_p := \log_R p, \qquad M_\xi(D) := \max(\{u_p : p \mid D,\ p > Y\} \cup \{0\}). \tag{4.4}$$

For a nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$, define

$$H_{\phi,\xi}(D) := \max_{\substack{p \mid D \\ p > Y}} \left(\sum_{\substack{q \mid D \\ q > p}} u_q + u_p + \phi(u_p) \right), \tag{4.5}$$

with empty maximum zero. For $m \geq 1$, write $H_{m,\xi}$ when $\phi(u) = (m-1)u$. Deleting primes decreases s_D, $M_\xi(D)$ and $H_{\phi,\xi}(D)$, so their upper-bound conditions are divisor-closed. The same notation applies to fragment configurations, using the inclusive tail at each witness as in [14, §1.3].

Lemma 4.1 (Lower-order transfer)**.** *Let $R > 1$, $\xi \geq 0$ and $Y := R^\xi$. Let D, E be positive squarefree integers with $s_D \leq S$ and $s_E \leq T$. Put $a := B - T$, $b := B - S$, and $Q := [D, E] > R^B$.*

(i) If $B + \xi \geq 2T$ and

$$s_D \leq a \qquad \text{or} \qquad H_{2,\xi}(D) \leq a + \xi,$$

then $Q \in \mathcal{D}^{(1)}(Y)$.

(ii) If $B + \xi \geq \max(2S - T, 2T - S)$ and

$$s_D \leq a \quad \text{or} \quad H_{2,\xi}(D) \leq a + \xi, \qquad s_E \leq b \quad \text{or} \quad H_{2,\xi}(E) \leq b + \xi,$$

then $Q \in \mathcal{D}^{(2)}(Y)$.

Proof. Since $Q > R^B$, the size bounds force $s_D > a$ and $s_E > b$. For $Y \geq 2$, exponentiating the remaining conditions gives Proposition 2.4 with $X = R^B$, $D_0 = R^S$ and $E_0 = R^T$.

If $1 \leq Y < 2$ and $D > 1$, its least prime factor gives $H_{2,\xi}(D) > s_D + \xi > a + \xi$, a contradiction. Thus $D = 1$. In (ii), the same argument gives $E = 1$. In (i), writing $X = R^B$ and $E_0 = R^T$, we have $Q = E > X$ and $E^2 \le E_0^2 \le XY < EY$, so $E < Y < 2$ and again $E = 1$. Hence $Q = 1$, which belongs to every dense class. $\square$

For order three, put

$$a \coloneqq B - T, \qquad b \coloneqq B - S, \qquad A \coloneqq a + \eta_D, \qquad C \coloneqq b + \eta_E, \qquad A + C \le B + \xi. \tag{4.6}$$

Proposition 4.2 (Transfer to order three)**.** *Let D, E be positive squarefree integers with $s_D \le S$ and $s_E \le T$. Suppose the nonnegative nondecreasing functions ϕ_D, ϕ_E satisfy*

$$\phi_D(u) + \phi_E(u) = 3u \qquad (u \ge 0). \tag{4.7}$$

Define

$$\mathscr{O}_\phi \coloneqq \{s_D \le a\} \cup \{H_{\phi_D,\xi}(D) \le A,\ \phi_E(M_\xi(D)) \le C\}, \tag{4.8}$$
$$\mathscr{I}_\phi \coloneqq \{s_E \le b\} \cup \{H_{\phi_E,\xi}(E) \le C,\ \phi_D(M_\xi(E)) \le A\}. \tag{4.9}$$

If $D \in \mathscr{O}_\phi$, $E \in \mathscr{I}_\phi$, and $Q \coloneqq [D, E] > R^B$, then $Q \in \mathcal{D}^{(3)}(Y)$.

Proof. The size bounds again force $s_D > a$ and $s_E > b$. Set $f(t) \coloneqq R^{\phi_D(\log_R t)}$ and $g(t) \coloneqq R^{\phi_E(\log_R t)}$. The remaining support conditions exponentiate to (2.1), with caps R^A, R^C whose product is at most $R^B Y$. Proposition 2.3 therefore applies for $Y \ge 2$. If $1 \le Y < 2$ and $Q > 1$, its primewise argument at the least prime $p \mid Q$ gives $p^3 < Y$, a contradiction. Thus $Q = 1$ in that case. $\square$

For every order-three row, use

$$\phi_D(u) \coloneqq \min\left\{\frac{3}{2}u, L\right\}, \qquad \phi_E(u) \coloneqq 3u - \phi_D(u), \qquad L \coloneqq \frac{23}{40}C. \tag{4.10}$$

These functions are nondecreasing and satisfy (4.7). Both opposite-root conditions in (4.8)–(4.9) remain part of the support.

4.2.2. *The retained distribution estimates.* Return to $R = x^{\rho_*}$. Use the old prime and new minorant source ladders, and their retained rows, from [14, §1.3], with

$$\tau \coloneqq 10^{-10}, \qquad \sigma_{m,\tau} \coloneqq \frac{1}{2} - a_* + \tau. \tag{4.11}$$

The strict distribution inequalities, lower-order transfer guards, order-three budgets and coverage of all mixed moduli are verified in [14, Lem. 1.1].

Let $\mathscr{O} \subseteq \mathscr{H}_O$ impose every retained outer predicate of both ladders. Write $\mathscr{L}_{\text{old}}$ and $\mathscr{L}_{\text{new}}$ for their raw inner predicates, including the radial bounds and global fragment cap, before imposing the cell restrictions. The actual inner domains are

$$\begin{aligned} \mathscr{L}_1 &\coloneqq \mathscr{H}_1 \cap \mathscr{L}_{\text{new}}, \\ \mathscr{L}_0 &\coloneqq \mathscr{H}_0 \cap \mathscr{L}_{\text{old}} \cap \mathscr{L}_{\text{new}}. \end{aligned} \tag{4.12}$$

Thus $\mathscr{L}_0 \subseteq \mathscr{L}_1$; the raw old and new predicates need not be nested. All source predicates are divisor-closed, so coefficient roots inherit their restrictions.

4.2.3. *Distribution estimates for inner squares.* Since $\rho_* T_1 = 251/1000 > 1/4$, pairs of inner roots require a source beyond Bombieri–Vinogradov for both the prime indicator and the minorant. For roots of normalized radius at most T, put $c \coloneqq B - T$ and define

$$\mathscr{S}(B, T, \xi) \coloneqq \{s \le c\} \cup \{H_{2,\xi} \le c + \xi\}. \tag{4.13}$$

Lemma 4.3 (Transfer for two inner roots)**.** *Suppose $\xi \ge 0$, $c + \xi \ge 0$, and the squarefree roots $D, E \in \mathscr{S}(B, T, \xi)$ have normalized radii at most T. If $Q \coloneqq [D, E] > R^B$, then $Q \in \mathcal{D}^{(2)}(R^\xi)$.*

Proof. Apply Lemma 4.1(ii) with both radius bounds equal to T. Its guard is $B + \xi \ge T$, equivalently $c + \xi \ge 0$. $\square$

Put

$$B_{\mathrm{BV}} := \frac{1}{2\rho}, \qquad c_s := B_{\mathrm{BV}} - T_1, \tag{4.14}$$

and take

$$\sigma := \frac{100001}{10^6}, \qquad \omega_s := \frac{31}{10000}, \qquad \delta_s := \frac{21319}{800000}, \qquad \xi_s := \delta_s/\rho. \tag{4.15}$$

At (ω_s, δ_s), the prime estimate with this σ and the minorant estimate with $\sigma_{m,\tau}$ have strict margins, and

$$\frac{1}{2} + 2\omega_s - 2\rho T_1 = \frac{55122259}{13124945000} > 0, \qquad c_s + \xi_s > 0; \tag{4.16}$$

see [14, Lem. 1.1].

The retained new order-two row 12 imposes $\{s \le b_{\mathrm{n},12}\} \cup \{H_{2,\xi_{\mathrm{n},12}} \le b_{\mathrm{n},12} + \xi_{\mathrm{n},12}\}$. The same lemma gives

$$c_s - b_{\mathrm{n},12} > \frac{4}{100}, \qquad \xi_s - \xi_{\mathrm{n},12} > \frac{5}{100}. \tag{4.17}$$

Increasing the activation removes witnesses, so this row implies $\mathscr{S}(B_{\mathrm{BV}}, T_1, \xi_s)$. Lemma 4.3 supplies the common order-two source above $R^{B_{\mathrm{BV}}} < x^{1/2}$, while Bombieri–Vinogradov applies below it. Since $\xi_s > 0$ and the roots are coprime to W, Lemma 2.2 permits adjoining W; the strict base and level bounds absorb $W = x^{o(1)}$.

4.2.4. *Finite-band approximation.* Fix the source rows and their positive activations, and put $\xi_{\min} := \min_\ell \xi_\ell > 0$. Keep the radial shell boundaries fixed, and partition $[\xi_{\min}, \zeta]$ into finitely many bands, retaining every activation, cap and breakpoint of (4.10) that lies in this interval. In each coordinate, combine the primes in a band into one block. At tier ℓ, primes at most R^{ξ_ℓ} form a smooth seed. For each root, order the remaining block logarithms $z_1 \le \cdots \le z_M$ and put

$$H^{\mathrm{band}}_{\phi,\xi_\ell} := \max_{1 \le j \le M} \left(\sum_{v>j} z_v + z_j + \phi(z_j) \right), \tag{4.18}$$

again with empty maximum zero. The seed contributes to the radial total and smaller-prime prefixes, but is not a witness.

Lemma 4.4 (Fixed-band sources)**.** *Replace the prime witnesses and activated maxima in each source predicate by the band blocks. The resulting supports are divisor-closed and imply the same dense-divisibility conclusion on that row's modulus band.*

Proof. Deleting a prime decreases the ordered block logarithms, hence their tail sums, maxima and radial total. For an activated prime p, let j be the first block containing a prime at least p. Then $z_j \ge u_p$, and the seed and preceding blocks contain only primes below p. A block-tail bound with threshold A therefore gives

$$\phi(u_p) \le \phi(z_j) \le A - s_D + \log_R D_{<p}.$$

The block maximum also bounds every activated prime. Thus the primewise tail and opposite-root conditions hold, and the transfer results above apply. □

For large x, all prime divisors of W lie below every activation. The roots are coprime to W, so Lemma 2.2 preserves their source class after adjoining W. Moreover,

$$\rho_* = \frac{2624989}{10^7} < \rho = \frac{262499}{10^6}. \tag{4.19}$$

For a row with upper boundary $B_\ell^+ = (1/2 + 2\omega_\ell)/\rho$, this gives

$$R^{\xi_\ell} = x^{(\rho_*/\rho)\delta_\ell} < x^{\delta_\ell}, \qquad W[D,E] \le x^{\rho_* B_\ell^+ + o(1)} < x^{1/2 + 2\omega_\ell - \varepsilon_\ell}$$

for some fixed $\varepsilon_\ell > 0$. The source estimates therefore apply at each fixed band resolution, with the fixed Fourier subdivisions, divisor weights and presieving modulus.

As the band mesh tends to zero, the probability that two fragments share a coordinate and band is at most

$$\frac{k}{2}\sum_{v}\left(\int_{\alpha_v}^{\beta_v}\frac{\mathrm{d}u}{u}\right)^2 \longrightarrow 0. \tag{4.20}$$

This follows from the second Poisson factorial moment and the finite intensity above $\xi_{\min}$. Multiplication by $(e^{\gamma}\zeta)^k\|F\|_\infty^2$ bounds the corresponding weighted ν^k-mass. Away from collisions, each active block is one fragment, so all source predicates agree. The smooth seed retains its full mass.

Each band law has an atom at zero and an absolutely continuous positive part. On every zero-band stratum, radial, activation and nonconstant support boundaries are null. For ϕ_D, a cap at or above its plateau is vacuous; a smaller cap has one boundary. One-sided smooth extensions retain the zero atoms, and removing thin neighborhoods of the remaining boundaries gives inward approximations within the hereditary supports.

Dominated convergence and (3.21) give convergence of the ordinary and face forms, including changing inner restrictions. The positive Palm and factorial integrals also converge: on bounded radial supports, the mark counts above each fixed cutoff and the fixed Chernoff factors are bounded. Fix the strict source, cap and counting margins, then choose finitely many bands and inward smooth approximations fine enough to preserve them. Let $x \to \infty$ with these choices fixed. No uniformity in the number of bands or in the smoothing derivatives is required.

4.3. **A hybrid Selberg inequality.** Retain $m_{\mathrm{true}} = 1 - \kappa_{\mathrm{true}}$ from Proposition 2.21, and put

$$\kappa_{\mathrm{def}} := \frac{1}{50000}, \qquad m_0 := 1 - \kappa_{\mathrm{def}} = \frac{49999}{50000}. \tag{4.21}$$

Thus $m_{\mathrm{true}} \geq m_0$ and $\kappa_{\mathrm{true}} \leq \kappa_{\mathrm{def}}$. Fix a detected coordinate i. For real outer, inner and correction sums $A = A_F$, B_0 and C, respectively, write $D = A - B_0$, $P = P(n + h_i)$, $\varrho = \varrho(n + h_i; x)$ and $\mathfrak{b} = \mathfrak{b}(n + h_i; x)$.

For $\eta > 0$, expansion using $A = B_0 + D$ and $\varrho = P - \mathfrak{b}$ gives

$$\begin{aligned} &PA^2 - \left[P(2AB_0 - B_0^2) + 2\varrho DC - PC^2 - \eta\mathfrak{b}D^2 - \eta^{-1}\mathfrak{b}C^2\right] \\ &\quad = P(D - C)^2 + \frac{\mathfrak{b}}{\eta}(\eta D + C)^2 \geq 0. \end{aligned} \tag{4.22}$$

since $P, \mathfrak{b} \geq 0$.

Let F be a real source-supported outer diagonal profile and let $\mathscr{L}_0 \subseteq \mathscr{L}_1$ be complete hereditary inner domains, including their physical cell restrictions. With $V_i = E_i F$, put

$$I := \|F\|_2^2, \qquad J_{0,i} := \|\mathbf{1}_{\mathscr{L}_0} V_i\|_2^2, \tag{4.23}$$

$$J_{+,i} := \|\mathbf{1}_{\mathscr{L}_1 \setminus \mathscr{L}_0} V_i\|_2^2, \qquad E_{0,i} := \|V_i\|_2^2 - J_{0,i}. \tag{4.24}$$

Theorem 4.5 (Signed minorant criterion). *Assume $I > 0$ and the following conditions:*

(1) every actual outer–$\mathscr{L}_0$ modulus has a prime-distribution estimate for $P = \mathbf{1}_{\mathbb{P}}$, as in Proposition 2.12(iii);
(2) every actual outer–$\mathscr{L}_1$ modulus has the fixed-minorant estimate;
(3) every actual self-pair modulus from two $\mathscr{L}_1$ roots has both estimates;
(4) the global cap (3.27)*, ordinary outer-square counting and Proposition 3.11 apply.*

If

$$\frac{\rho_*}{I}\sum_{i=1}^{k}\left[J_{0,i} + \left(m_0\sqrt{J_{+,i}} - \sqrt{\kappa_{\mathrm{def}} K_{\mathrm{ex}} E_{0,i}}\right)_+^2\right] > 1, \tag{4.25}$$

then DHL$[k, 2]$ *holds.*

Proof. By Lemma 3.8 and §4.2.4, choose B_0 with profile $H_0 = \mathbf{1}_{\mathscr{L}_0} V_i$ and C with profile H supported on $\mathscr{L}_1 \setminus \mathscr{L}_0$, up to arbitrarily small fixed approximation errors. The coefficient roots of C remain in the hereditary domain $\mathscr{L}_1$, although its profile support need not be hereditary. Bands and inward smoothing are fixed before $x \to \infty$. On the support of $\mathfrak{b}$, the cap gives $D = A^0_{F,i} - B_0$ by (3.28); its limiting squared norm is $E_{0,i}$ by (3.23). For the correction square, use

$$\mathfrak{b}C^2 = PC^2 - \varrho C^2. \tag{4.26}$$

Both distribution estimates apply to the same actual self-pair moduli. Using (3.1), normalize by

$$\mathcal{M}_x(U) := (\rho_*^{-(k-1)} C_x)^{-1} \sum_{\substack{x \le n < 2x \\ n \equiv b_0 \pmod{W}}} U(n).$$

The source hypotheses, (4.26) and Proposition 3.11 give

$$\begin{aligned}
\mathcal{M}_x(P(2AB_0 - B_0^2)) &= J_{0,i} + o(1), \\
\mathcal{M}_x(2\varrho DC) &= 2m_{\mathrm{true}} \langle V_i - H_0, H \rangle + o(1), \\
\mathcal{M}_x(PC^2) &= \|H\|_2^2 + o(1), \\
\mathcal{M}_x(\mathfrak{b}C^2) &= \kappa_{\mathrm{true}} \|H\|_2^2 + o(1), \\
\mathcal{M}_x(\mathfrak{b}D^2) &\le K_{\mathrm{ex}} E_{0,i} + o(1).
\end{aligned}$$

For fixed $\eta > 0$, optimizing scalar multiples of $\mathbf{1}_{\mathscr{L}_1 \setminus \mathscr{L}_0} V_i$ gives the increment

$$\frac{m_{\mathrm{true}}^2 J_{+,i} \eta}{\eta + \kappa_{\mathrm{true}}} - \eta K_{\mathrm{ex}} E_{0,i}.$$

Its supremum over $\eta > 0$, including zero correction, is

$$\left(m_{\mathrm{true}} \sqrt{J_{+,i}} - \sqrt{\kappa_{\mathrm{true}} K_{\mathrm{ex}} E_{0,i}}\right)_+^2 \ge \left(m_0 \sqrt{J_{+,i}} - \sqrt{\kappa_{\mathrm{def}} K_{\mathrm{ex}} E_{0,i}}\right)_+^2.$$

The ordinary square has main term $\rho_*^{-k} C_x I$. Thus (4.25) and (4.22) make the prime-detection sum (1.5) positive after the fixed approximation errors are absorbed. Since $A(n)^2 \ge 0$, this proves DHL$[k, 2]$ for each fixed admissible k-tuple; compare [16, Lem. 3.4]. $\square$

For $u, v \ge 0$ and $0 < \lambda < m_0$,

$$(m_0 \sqrt{u} - \sqrt{v})_+^2 \ge (m_0^2 - m_0 \lambda) u + (1 - m_0/\lambda) v. \tag{4.27}$$

Indeed, if $v \ge m_0^2 u$, the right side is nonpositive; otherwise, the difference is $m_0(\sqrt{\lambda u} - \sqrt{v/\lambda})^2 \ge 0$. Set

$$\begin{aligned}
\lambda &:= \frac{1}{125}, \qquad a_h := m_0^2 - m_0 \lambda, \\
b_h &:= (1 - m_0/\lambda) \kappa_{\mathrm{def}} K_{\mathrm{ex}}, \qquad d_0 := 1 - a_h - b_h.
\end{aligned} \tag{4.28}$$

These satisfy $d_0 > 0$, $0 < a_h + b_h < 1$ and $-1/1000 < b_h < 0$. Applying (4.27) with $u = J_{+,i}$ and $v = \kappa_{\mathrm{def}} K_{\mathrm{ex}} E_{0,i}$ gives the lower form

$$J_{0,i} + a_h J_{+,i} + b_h E_{0,i} = d_0 J_{0,i} + a_h J_{1,i} + b_h J_{\mathrm{full},i}, \tag{4.29}$$

where $J_{1,i} = J_{0,i} + J_{+,i}$ and $J_{\mathrm{full},i} = \|V_i\|_2^2$ uses the unrestricted face norm.

4.4. **Restoring the arithmetic supports.** Use the caps of §4.1 and the actual inner domains (4.12). Work in $L^2(\mathscr{H}_O,\nu^k)$, extending functions by zero. Let E_i integrate coordinate i, let P_O multiply by $\mathbf{1}_{\mathscr{O}}$, and put $H_{a,i}=\mathbf{1}_{\mathscr{H}_a}$ and $R_{a,i}=\mathbf{1}_{\mathscr{L}_a}$ on face i, for $a=0,1$. Define

$$\mathcal{A} \coloneqq \sum_i E_i^*(d_0H_{0,i}+a_hH_{1,i}+b_h)E_i, \tag{4.30}$$

$$\mathcal{B} \coloneqq \sum_i E_i^*(d_0R_{0,i}+a_hR_{1,i}+b_h)E_i, \tag{4.31}$$

$$\mathcal{D} \coloneqq \mathcal{A}-\mathcal{B} = \sum_i E_i^*\big(d_0(H_{0,i}-R_{0,i})+a_h(H_{1,i}-R_{1,i})\big)E_i \geq 0. \tag{4.32}$$

The negative full-face term cancels from $\mathcal{D}$.

For a retained total $s_{\widehat{i}}=\sum_{j\neq i}t_j<S$, put $w_i=S-s_{\widehat{i}}+(k-1)t_i$. Since the total-size density is $\rho_{\mathrm{D}}(t_i/\zeta)\leq 1$,

$$\int_{\mathscr{H}_O(\mathbf{X}_{\widehat{i}})}\frac{\mathrm{d}\nu(X_i)}{w_i} \leq \int_0^{S-s_{\widehat{i}}}\frac{\mathrm{d}t_i}{S-s_{\widehat{i}}+(k-1)t_i}=\frac{\log k}{k-1}.$$

The boundary $s_{\widehat{i}}=S$ has zero fibre measure. Weighted Cauchy–Schwarz and $\sum_i w_i=kS$ give $\sum_i E_i^*E_i\leq Sk\log(k)/(k-1)$. For $k=40$, the inequality $\sum_{j=0}^{11}(369/100)^j/j!>40$ gives $\log 40<369/100$. Together with $S=2742997/2624989<130/123$, this proves

$$\sum_i E_i^*E_i \leq C_{\mathrm{op}}\,\mathrm{Id}, \qquad C_{\mathrm{op}}\coloneqq 4. \tag{4.33}$$

The multipliers of $\mathcal{B}$ are $1, a_h+b_h, b_h$, so

$$-|b_h|C_{\mathrm{op}}\,\mathrm{Id}\leq \mathcal{B}\leq C_{\mathrm{op}}\,\mathrm{Id}\,. \tag{4.34}$$

For F in this space and $V_i=E_iF$, put

$$e=(1-P_O)F, \qquad I_H=\|F\|_2^2, \qquad \alpha=\|e\|_2^2, \qquad J_{\lambda,H}=\langle F,\mathcal{A}F\rangle.$$

All forms here use unscaled products of ν. The inner deletion is

$$\beta\coloneqq\langle F,\mathcal{D}F\rangle = d_0\sum_i\int_{\mathscr{H}_0\setminus\mathscr{L}_0}|V_i|^2\,\mathrm{d}\nu^{k-1}+a_h\sum_i\int_{\mathscr{H}_1\setminus\mathscr{L}_1}|V_i|^2\,\mathrm{d}\nu^{k-1}. \tag{4.35}$$

Define its two uncombined losses by

$$\beta_{\mathrm{old}}\coloneqq\sum_i\int_{\mathscr{H}_0\setminus\mathscr{L}_{\mathrm{old}}}|V_i|^2\,\mathrm{d}\nu^{k-1}, \qquad\qquad \beta_{\mathrm{new}}\coloneqq\sum_i\int_{\mathscr{H}_1\setminus\mathscr{L}_1}|V_i|^2\,\mathrm{d}\nu^{k-1}. \tag{4.36}$$

The intersection (4.12) and $\mathscr{H}_0\subseteq\mathscr{H}_1$ imply

$$\beta\leq d_0\beta_{\mathrm{old}}+(d_0+a_h)\beta_{\mathrm{new}}=d_0\beta_{\mathrm{old}}+(1-b_h)\beta_{\mathrm{new}}. \tag{4.37}$$

On face i, write $\mathbf{Y}=\mathbf{X}_{\widehat{i}}$ and put

$$h_i(\mathbf{Y})\coloneqq\begin{cases}1, & \mathbf{Y}\in\mathscr{H}_0,\\ \max\{|a_h+b_h|,|b_h|\}, & \mathbf{Y}\in\mathscr{H}_1\setminus\mathscr{H}_0,\\ |b_h|, & \mathbf{Y}\notin\mathscr{H}_1.\end{cases} \tag{4.38}$$

Then $m_i=d_0R_{0,i}+a_hR_{1,i}+b_h$ satisfies $|m_i|\leq h_i$. For a measurable cover $\mathscr{H}_O\setminus\mathscr{O}\subseteq\bigcup_j\mathscr{B}_j$ and positive rational c_j, define the nonnegative outer bound

$$\mathcal{E}_O\coloneqq\sum_{i,j}\int_{\mathscr{B}_j}h_i(\mathbf{Y})\left(c_j|F(\mathbf{X})|^2+\frac{|V_i(\mathbf{Y})|^2}{c_j}\right)\mathrm{d}\nu^k(\mathbf{X}). \tag{4.39}$$

Proposition 4.6 (Restoration of the quadratic form)**.** *One has*

$$\rho_*\langle P_O F, \mathcal{B}(P_O F)\rangle - \|P_O F\|_2^2 \geq \rho_*(J_{\lambda,H} - \beta - \mathcal{E}_O) - I_H + (1 - \rho_*|b_h|C_{\mathrm{op}})\alpha. \tag{4.40}$$

If F is real and $P_O F$ with the stated inner domains satisfies the realization and source hypotheses of Theorem 4.5, then DHL$[k,2]$ *holds if*

$$\rho_*\left(J_{\lambda,H}^- - \mathcal{E}_O^+ - d_0\beta_{\mathrm{old}}^+ - (1-b_h)\beta_{\mathrm{new}}^+\right) > I_H^+, \tag{4.41}$$

where superscripts denote outward lower and upper bounds.

Proof. Since $P_O F = F - e$, $\mathcal{B} = \mathcal{A} - \mathcal{D}$ and $\|P_O F\|_2^2 = I_H - \alpha$,

$$\begin{aligned}\rho_*\langle P_O F, \mathcal{B}(P_O F)\rangle - \|P_O F\|_2^2 = \rho_* J_{\lambda,H} - I_H - \rho_*\beta& \\ - 2\rho_* \operatorname{Re}\langle e, \mathcal{B}F\rangle + \rho_*\langle e, \mathcal{B}e\rangle + \alpha&.\end{aligned}$$

Using $e = \mathbf{1}_{\mathscr{H}_O\setminus\mathscr{O}}F$, $|m_i| \leq h_i$, and the cover, Young's inequality gives

$$2\operatorname{Re}\langle e, \mathcal{B}F\rangle \leq \sum_{i,j}\int_{\mathscr{B}_j} 2h_i(\mathbf{Y})|F(\mathbf{X})V_i(\mathbf{Y})|\,\mathrm{d}\nu^k \leq \mathcal{E}_O.$$

Also (4.34) gives $\langle e, \mathcal{B}e\rangle \geq -|b_h|C_{\mathrm{op}}\alpha$, proving (4.40). Since $\rho_* < 1/3$ and $|b_h| < 1/1000$, the coefficient of α is positive. Thus (4.37) and (4.41) make the restored quadratic form positive, so $P_O F \neq 0$. Apply (4.29) to $E_i(P_O F)$ and then Theorem 4.5. ☐

4.5. **Support-error bounds.** The finite source comparisons and the positive cover of their failures are given in [14, §§1.3, 1.5]. We record the consequence needed for support restoration.

Proposition 4.7 (A positive cover of the support failures)**.** *For the trial and source domains of §4.1, all support failures are covered by the six classes*

$$O_2, \qquad O_{5/2}, \qquad I_2^{\mathrm{old}}, \qquad I_{5/2}^{\mathrm{old}}, \qquad I_2^{\mathrm{new}}, \qquad I_{5/2}^{\mathrm{new}}. \tag{4.42}$$

The outer classes have dimension 40*, the inner classes dimension* 39*, and the subscript is the effective obstruction order. Both inner pairs are imposed on the base domain. Their indicators admit a positive cover with* 61 *low-witness intervals,* 30 *rank-two intervals and six higher-rank terms, including coincident marked coordinates and every absorbed lower-order condition.*

Proof. The rational comparisons in [14, Lem. 1.1] verify the actual-LCM bands, inward cell caps, activations, and largest-fragment and opposite-root bounds. The lower-order transfer conditions give $H_{2,\xi}$ failures. For order three, the allocation (4.10) satisfies $\max(A, C) \leq 7L/3$; after excluding largest witnesses, its failures are $H_{5/2,\xi}$ failures by [14, Lem. 1.4]. This gives the six classes.

For each class, split nonlargest witnesses at the point p_G of [14, §1.5]. Low witnesses are covered by its positive Palm–Chernoff bound. Above p_G, exactly two fragments give the rank-two terms, with the two marks in either the same or distinct coordinates; at least three give the third-factorial term. The half-open partitions in [14, §1.5.1] cover the full ranges, and [14, Lem. 1.6] shows that radial clipping loses no low witness, including those absorbed from lower-order rows. Every term is nonnegative, so overlap does not affect the upper bound. ☐

4.6. **Integral estimates and completion of the proof.** We use the unscaled forms I_H, $J_{\lambda,H}$, β_{old}, β_{new} and $\mathcal{E}_O$ from §4.4. For each outer component choose the rational Young parameter specified in [14, §2.4]. Let L^+ be the sum of the resulting outer bounds and the inner bounds weighted by d_0 and $1 - b_h$, respectively.

Lemma 4.8 (Numerical bounds)**.** *For the stated trial there are bounds* $0 < I_H^- \le I_H \le I_H^+$ *and* $J_{\lambda,H}^- \le J_{\lambda,H}$ *such that*

$$d_0\beta_{\text{old}} + (1-b_h)\beta_{\text{new}} + \mathcal{E}_O \le L^+, \tag{4.43}$$

$$\frac{\rho_* J_{\lambda,H}^-}{I_H^+} > \frac{500103}{500000}, \qquad \frac{L^+}{I_H^-} \le \frac{696075110}{10^{12}} < \frac{697}{10^6}. \tag{4.44}$$

Proof. These are the bounds of [14, Lem. 2.5]. Its signed cap form is $\langle F, \mathcal{A}F\rangle$, with face multipliers 1, $a_h + b_h$ and b_h, and its component bounds use the cover in Proposition 4.7. The companion divides every form and loss by the same positive factor. Multiplying its bounds by that factor gives the stated enclosures and leaves both quotients unchanged. □

Proof of Theorem 1.1. Fix an arbitrary admissible tuple $\mathcal{H} = \{h_1, \ldots, h_{40}\}$ of distinct integers. Since $I_H^+ \ge I_H^- > 0$, Lemma 4.8 gives

$$\frac{\rho_*(J_{\lambda,H}^- - L^+) - I_H^+}{I_H^+} > \frac{1}{50000}. \tag{4.45}$$

Corollary 2.19 and Proposition 2.22, with the source comparisons in [14, Lem. 1.1], supply the mixed estimates. The argument of §4.2.3 supplies both self-square estimates. These are uniform in a primitive global residue. Presieving and Proposition 2.14 give the required residue classes for this fixed tuple. The finite-band approximation of §4.2.4, with the bands and smoothing fixed before $x \to \infty$, verifies the realization hypotheses of Proposition 4.6. By (4.43), the strict inequality above makes its restored quadratic form positive. Hence the prime-detection sum is positive for all sufficiently large x, and $A(n)^2 \ge 0$ gives infinitely many translates of $\mathcal{H}$ containing two primes. Since $\mathcal{H}$ was arbitrary, DHL$[40, 2]$ follows. □

Appendix A. Analytic estimates

Write $\mathrm{e}(t) := \exp(2\pi i t)$ and $\mathrm{e}_q(t) := \mathrm{e}(t/q)$. All modular fractions retain their original unit domains.

A.1. **Divisor estimates.**

Lemma A.1 (Divisor bounds in progressions)**.** *Fix* $0 < \vartheta < 1$ *and the coefficient bounds. Uniformly for coefficient sequences* v *at polynomially bounded scales* N*, one has* $\sum_n |v(n)|^2 \ll N(\log x)^C$ *and* $\sum_{n\equiv a\,(q)} |v(n)| \ll_\varepsilon x^\varepsilon(N/q+1)$*. If a divisor-bounded sequence* u *is supported in* $[cx, Cx]$ *and in a fixed number of intervals of total length* $O(x(\log x)^{-D})$*, with coefficient bounds independent of* D*, then for fixed* J, A *and sufficiently large* D*,*

$$\sum_{q\le x^\vartheta} \tau(q)^J \max_{(a,q)=1} |\Delta(u; a \bmod q)| \ll_A x(\log x)^{-A}. \tag{A.1}$$

Proof. The first two bounds follow from [15, Lem. 1.3(i),(ii)] and the progression count $O(N/q+1)$. Enlarge each boundary interval to length $\asymp H = x(\log x)^{-D}$. Shiu's theorem [18, Thm. 1, (2.1)–(2.2)], applied to a fixed divisor-power majorant, bounds its progression mass and reduced-residue mean by $H\varphi(q)^{-1}(\log x)^C$. Its length conditions hold for $q \le x^\vartheta$ with $0 < \alpha < \min\{1/2, 1-\vartheta\}$; $q = 1$ has zero discrepancy. Now use $\sum_{q\le x^\vartheta} \tau(q)^J/\varphi(q) \ll (\log x)^{C'}$. □

Lemma A.2 (Zero frequency)**.** *Suppose* $(b_1b_2r, q_0) = 1$*, and let* $0 \le C(n,\ell) \le 1$ *be bounded by the indicator of* $(b_1 - b_2)n + b_1\ell r \equiv 0 \pmod{q_0}$*. For* $L \ge 1$ *and any finitely supported* β*,*

$$\sum_{|\ell|\ll L} \sum_n |\beta(n)\beta(n+\ell r)| C(n,\ell) \ll (1 + L/q_0) \sum_n |\beta(n)|^2. \tag{A.2}$$

In the dispersion reduction of [15, §5.3] *and* [19, Appendix], *the noncoprime zero-frequency contribution satisfies*

$$|\Sigma_0'| \ll MN(\log x)^C \left(1 + \frac{N}{RD_0}\right), \qquad D_0 := \exp((\log x)^{1/3}). \tag{A.3}$$

Its ratio to MN^2/R *is* $O((\log x)^C(R/N + D_0^{-1}))$.

Proof. Apply $2|uv| \le |u|^2 + |v|^2$. For fixed n, the congruence selects one class of $\ell \bmod q_0$; after $m = n+\ell r$, the second square uses $(b_1-b_2)m+b_2\ell r \equiv 0 \pmod{q_0}$, also one class. This proves (A.2), including any further unit restrictions. Insert $L \asymp N/R$ in [15, §5.3, preceding (5.26)]; harmonic summation over $q_0 \ge D_0$ and $\sum_n |\beta(n)|^2 \ll N(\log x)^C$ give (A.3). In our applications $R/N \ll x^{-2\varepsilon}$, giving every fixed logarithmic saving relative to MN^2/R. □

A.2. **Reciprocal phases.** Put $E_m(c;t) := \mathbf{1}_{(t,m)=1} \exp(2\pi i c\bar{t}/m)$ for $m \ge 2$, with $E_1(c;t) := 1$; the inverse is evaluated only on units.

Lemma A.3 (Reciprocal phases). *Let* d_1, d_2 *be squarefree and* $d \mid q$, *where* $q := [d_1, d_2]$ *is singly* y*-densely divisible. Suppose* $N \ge d$, $N \le q^{C_0}$, *and* ψ_N *is smooth at scale* N, *with fixed-order derivatives bounded by powers of* $\log(2qN)$. *Put* $\delta_i := d_i/(d_1,d_2)$, $\delta_i' := \delta_i/(d,\delta_i)$ *and* $B := \prod_{i=1}^2 (c_i,\delta_i')/\delta_i'$. *The sum*

$$S := \sum_{n \equiv a\,(d)} \psi_N(n) E_{d_1}(c_1; n+\ell_1) E_{d_2}(c_2; n+\ell_2)$$

satisfies, uniformly in the integer coefficients, shifts, and classes,

$$|S| \ll_\varepsilon q^\varepsilon \left(\sqrt{N/d}\,(qy)^{1/6} + (N/d)B\right), \tag{A.4}$$

$$|S| \ll_\varepsilon q^\varepsilon \left(\sqrt{q/d} + (N/d)B\right). \tag{A.5}$$

Proof. Put $X = N/d \ge 1$, $m = q/d$, and substitute $n = a + dv$. Primes dividing d contribute constants of modulus at most one. CRT leaves $F_p(v) = \mathbf{1}_{v \notin Z_p} \mathrm{e}_p(f_p(v))$ for $p \mid m$, where f_p is proper with at most two simple poles and $|Z_p| \le 2$; Z_p retains all original denominator zeros.

Apply [15, Prop. 4.12 and proof of Cor. 4.16] as follows. At nonzero-phase primes the masks change complete Fourier and translated-difference sums by at most two and four terms, respectively. Outside bounded primes, a translated difference vanishes exactly when $p \mid h$, for the shift difference h. At zero-phase primes, in the sum and its correlations, expand

$$\mathbf{1}_{v \notin Z} = 1 - \sum_{z \in Z} \mathbf{1}_{v \equiv z\,(p)}.$$

Each expansion has at most $5^{\omega(q)}$ terms and never increases the scale. For $m = rs$, the surviving phase factors satisfy $r_0 \mid r$, $s_0 \mid s$. On s_0, the off-diagonal bound remains $q^\varepsilon(\sqrt{s_0} + Y\sqrt{(s_0,h)}/\sqrt{s_0})$ at scale Y; the published gcd summation applies unchanged. Scales below one contribute $O(q^\varepsilon)$.

Each active exclusive prime has normalized mean $-1/p$; other local means have modulus at most one. The affine substitutions preserve the remaining means, giving B in each term. Completion and differencing yield

$$|S| \ll_\varepsilon q^\varepsilon(\sqrt{m} + XB),$$
$$|S| \ll_\varepsilon q^\varepsilon(\sqrt{X}(\sqrt{r} + s^{1/4}) + XB) \qquad (m = rs).$$

By [15, Lem. 2.10(i)], m is yd-densely divisible. Choose r as in Cor. 4.16; then $\sqrt{r}, s^{1/4} \le (myd)^{1/6} = (qy)^{1/6}$. All divisor and smoothness losses use the ambient q, since $N \le q^{C_0}$. □

In the applications of [15, §5], $d = q_0 \ll Q$ and $N/Q \gg x^{1/2-2\sigma-2\omega-\delta-O(\varepsilon)}$. The exponent margins for Type I of orders $1, 2$, Type II, and the minorant's smooth-factor estimate are, respectively, $1/10 + (98/5)\omega + 5\delta$, $26\omega + 7\delta$, $7/17 - O(\varepsilon)$, and $12\omega + 3\delta$; Type II uses $1/2 - 6\omega - \delta - O(\varepsilon)$. Thus N/q_0 grows polynomially, verifying $N \geq d$. Also $r \mid q$ with $r \gg Nx^{-\delta-O(\varepsilon)}$; the same inequalities give $N \leq q^3$.

A.3. **Hyper-Kloosterman correlations.** We use the normalization

$$\mathrm{Kl}_3(c; m) := \frac{1}{m} \sum_{\substack{x_1,x_2,x_3 \bmod m \\ x_1x_2x_3 = c \bmod m}} \mathrm{e}_m(x_1 + x_2 + x_3),$$

also for composite m. For a prime p, put $K_2(c; p) := \sum_{u \in \mathbb{F}_p^\times} \mathrm{e}_p(u + c/u)$; this second sum is unnormalized.

Lemma A.4 (Prime-field inputs)**.** *For every prime p and nonzero $c, A, B \in \mathbb{F}_p$,*

$$|\mathrm{Kl}_3(c; p)| \leq 3, \qquad \left| \sum_{t \in \mathbb{F}_p \setminus \{0,-1\}} K_2(A/t; p) K_2(B/(t+1); p) \right| \leq 8p\sqrt{p}. \tag{A.6}$$

Proof. The first estimate is [10, Thm. 4.1.1(1)–(2), p. 49]: rank three and weight two give the unnormalized bound $3p$. For the second, substitute $u \mapsto u^{-1}$ in [5, Prop. 2]. Its normalized sum is $K_2(c; p)/\sqrt{p}$, so its bound $8\sqrt{p}$ gives the displayed bound. The two excluded poles are retained in both statements. □

For $ab \neq 0$, the unit-masked correlation $C_p(a, b; h) := \sum_{v \in \mathbb{F}_p^\times} \mathrm{Kl}_3(av; p)\overline{\mathrm{Kl}_3(bv; p)}\mathrm{e}_p(hv)$ satisfies

$$|C_p(a, b; h)| \leq \begin{cases} p, & a = b,\ h = 0, \\ 9\sqrt{p}, & \text{otherwise.} \end{cases}$$

Indeed, orthogonality gives $C_p(a, b; 0) = p\mathbf{1}_{a=b} - 1 - p^{-1} - p^{-2}$. For $h \neq 0$, [5, Props. 2, 6] give $|C_p(a, b; h)| \leq 8\sqrt{p} + p^{-2}$ with this normalization; the unit mask removes $|\mathrm{Kl}_3(0; p)|^2 = p^{-2}$. Orthogonality and [10, Thm. 4.1.1, rank two] also bound the unit-masked Fourier coefficients of $v \mapsto \mathrm{Kl}_3(av; p)$ by $3\sqrt{p}$.

Lemma A.5 (Hyper-Kloosterman completion)**.** *Let s, r_1, r_2 be squarefree, $(s, r_1r_2) = 1$, and let a_i be a unit modulo r_is. Write $d := (r_1, r_2)$ and $u_i := r_i/d$. Put $q := s[r_1, r_2]$ and*

$$G := (a_2u_1^3 - a_1u_2^3, d)^{1/2}(a_2r_1^3 - a_1r_2^3, s)^{1/2}.$$

For $q, H \leq x^{C_0}$, a smooth weight w_H at any positive scale H, and every $\varepsilon > 0$,

$$\left| \sum_{(h,q)=1} w_H(h)\, \mathrm{Kl}_3(a_1h; r_1s)\overline{\mathrm{Kl}_3(a_2h; r_2s)} \right| \ll x^\varepsilon (1 + H/q)\sqrt{q}\, G. \tag{A.7}$$

The fixed-order derivatives of w_H are bounded by powers of $\log x$, as in the Type III completion of [15, §§6–7].

Proof. Put $s' = sd$; then s', u_1, u_2 are pairwise coprime, $r_is = u_is'$, and $q = s'u_1u_2$. Apply [15, Lem. 6.25] to (s', u_1, u_2). Since $(u_1, u_2) = 1$, every unit-masked complete Fourier coefficient is

$$\ll_\eta q^{1/2+\eta}(a_2u_1^3 - a_1u_2^3, s')^{1/2} = q^{1/2+\eta}G.$$

The equality uses $a_2r_1^3 - a_1r_2^3 = d^3(a_2u_1^3 - a_1u_2^3)$ and $(s, d) = 1$. For $H \geq 1$, [15, Lem. 4.9(i)] gives completion cost $O((1 + H/q)(\log x)^C)$; absorb its logarithmic loss and q^η in x^ε. For $H < 1$, use the $O(1)$ support size and [15, Rem. 6.10]. The case $q = 1$ is immediate. □

Lemma A.6 (A gcd average)**.** *Let $D \geq 1$ and $c_1, c_2 \geq 0$ be integers with $(c_1, D) = 1$, and let I, J be integer intervals of cardinalities L_I, L_J, respectively. Then*

$$\sum_{m \in I} \sum_{n \in J} (D, c_1 m - c_2 n)^{1/2} \leq \tau(D)(L_I L_J + L_J \sqrt{D}).$$

Proof. For each n and $k \mid D$, $(c_1, k) = 1$ gives at most $L_I/k + 1$ solutions of $c_1 m \equiv c_2 n \pmod{k}$. Thus

$$\sum_{m,n} (D, c_1 m - c_2 n)^{1/2} \leq L_J \sum_{k \mid D} \sqrt{k}(L_I/k + 1) \leq \tau(D)(L_I L_J + L_J \sqrt{D}).$$

□

In [15, §7.3, pp. 90–93], $a_i = A\overline{m_i}$ for the unit $A := ab_1 b_2 b_3 \bar{b}^3$. Keeping the original unit conditions, put

$$\Delta := m_1 u_1^3 - m_2 u_2^3, \qquad T := m_1 r_1^3 - m_2 r_2^3 = d^3 \Delta.$$

Modulo d and s, cancelling $A\overline{m_1 m_2}$ gives $G = (d, \Delta)^{1/2}(s, T)^{1/2}$. The integer diagonal remains $T = 0$; off it, T is independent of s and $\sum_{1 \leq s \leq K} (T, s)^{1/2} \leq K\tau(|T|)$. Since $(u_i, d) = 1$, Lemma A.6 gives

$$\sum_{m_1, m_2 \asymp M} (d, m_1 u_1^3 - m_2 u_2^3)^{1/2} \ll x^{o(1)}(M^2 + M\sqrt{d}).$$

After cancellation, omit the unit and off-diagonal restrictions in this nonnegative average and apply the divisor sums of [15, Lem. 7.4]. The replacement $s' = sd$ occurs only in the correlation estimate, leaving the scale S and dense divisor unchanged. Together with Proposition 2.15 and Lemmas A.2 and A.3, these estimates give the application of [15, §7].

A.4. **Triply densely divisible moduli.**

A.4.1. *Double completion.*

Lemma A.7 (Double completion)**.** *Let $q \mid m \leq x^C$, with m squarefree, and let A, B, L be residue classes modulo m. Let w_D, w_N be smooth weights at positive scales $D, N \leq x^C$, with arbitrary translations and fixed support and logarithmic derivative bounds. Put $F_1 = n + Bd$ and $F_2 = n + (B + L)d$. Uniformly in d_*, n_* mod q,*

$$\left| \sum_{\substack{d \equiv d_* \ (q) \\ n \equiv n_* \ (q)}} w_D(d) w_N(n) \mathbf{1}_{(F_1 F_2, m) = 1} \mathrm{e}_m\left(\frac{AL}{F_1 F_2}\right) \right| \ll_{C,\varepsilon} \frac{x^\varepsilon (AL, m)}{q} \left(m + N + D + \frac{ND}{m}\right). \tag{A.8}$$

There is no lower bound on $D/q, N/q$ or density assumption on m.

Proof. For $q = 1$, allow affine constants in both factors. At $p \nmid AL$, their invertible affine change gives the complete sum $\sum_{u,v \in \mathbb{F}_p^\times} \mathrm{e}_p(AL/(uv) + au + bv)$. Lemma A.4 bounds it by $3p$ for $ab \neq 0$; direct summation gives at most p otherwise. For $p \mid AL$ use p^2 with the original unit mask. CRT therefore bounds every Fourier coefficient by $3^{\nu(m)} m(AL, m)$, where $\nu(m)$ counts its prime factors. Poisson summation and two integrations by parts give normalized Fourier ℓ^1 cost $O((1 + X/m)(\log x)^{O(1)})$ at every scale $X > 0$. Applying this in both variables proves the affine estimate.

For $q \mid m$, substitute $d = qd' + d_*$, $n = qn' + n_*$ and write $m = qm_1$. The q-part is constant of modulus at most one, and zero if an original denominator is a nonunit. The m_1-part has the same slopes, affine constants, and numerator AL/q^3 mod m_1. Apply the affine estimate at scales $D/q, N/q$, using $(AL/q^3, m_1) = (AL, m_1) \leq (AL, m)$; $m_1 = 1$ is immediate. This is the congruence reduction of [19, Lem. 13]. □

A.4.2. *The dispersion argument.*

Proof of Proposition 2.18. Write ω_0, δ_0 for the given parameters and enlarge them to ω, δ preserving (2.14). Choose $0 < \varepsilon < 10^{-100}\delta$ sufficiently small. First suppose $N = x^\gamma$ with

$$\max\left\{\frac{1}{4} + 12\omega + 4\delta + 100\varepsilon,\ 32\omega + 10\delta + 400\varepsilon\right\} \le \gamma \le \frac{1}{2} - 4\omega - 2\delta - 50\varepsilon. \tag{A.9}$$

Use the sums $\Sigma_1, \ldots, \Sigma_6$ and parameter partitions of [19, Lems. 3–9], writing R_0 for its R and fixing coefficient supports and derivative bounds before taking suprema. The sums $\Sigma_3, \ldots, \Sigma_6$ retain the mask $\mathbf{1}_{((n+Bd)(\widetilde{n}+Bd),m)=1}$, with the integer substitution for $\widetilde{n}$ in Σ_6.

Strong order-three density with orientation $(1,1)$ factors each modulus as qr, with both factors singly x^{δ_0}-dense and $x^{-\delta_0-3\varepsilon}N \le r \le x^{-3\varepsilon}N$. By [15, §5.3, Cor. 5.3] and Lemma A.1, discard moduli whose part on primes at most $D_0 = \exp((\log x)^{1/3})$ exceeds $\exp((\log x)^{2/3})$, and move the remaining such primes from q to r. The transferred factor is $c = x^{o(1)}$; [15, Lem. 2.10(i)] makes both factors singly cx^{δ_0}-dense, hence x^δ-dense. Dyadic subdivision gives

$$x^{-\delta-4\varepsilon}N \ll R_0 \ll x^{-2\varepsilon}N, \qquad x^{1/2-\varepsilon} \ll R_0 Q \ll x^{1/2+2\omega+\varepsilon}. \tag{A.10}$$

Proposition 2.15 covers smaller moduli. After the dispersion square, put $q^{(i)} = q_0 q_i$ with $q_0 = (q^{(1)}, q^{(2)})$. Every $q_0 > 1$ exceeds D_0; Lemma A.2 applies on $n, n + \ell r \asymp N$ since $R_0/N \ll x^{-2\varepsilon}$.

Put $H = x^\varepsilon R_0 Q^2/(q_0 M)$; $H < 1$ contributes nothing. The quotient q_1 is singly $q_0 x^\delta$-dense. Choosing its divisor at target $X = x^{-5\varepsilon}Q/H$, or taking the whole quotient when $X > q_1$, gives $q_1 = u_1 v_1$ and

$$\begin{aligned} q_0^{-1}x^{-\delta-5\varepsilon}Q/H \ll U \ll x^{-5\varepsilon}Q/H, \\ q_0^{-1}x^{5\varepsilon}H \ll V \ll x^{\delta+5\varepsilon}H, \qquad\qquad UV \asymp Q/q_0. \end{aligned} \tag{A.11}$$

Here (A.9) gives $X \gg q_0 x^{2\omega+2\delta+43\varepsilon} > 1$. These are the choices in [15, §8.1, (8.3)–(8.5)], with the factor q_0^{-1} retained in the lower bound for V.

Set $D = N/(x^{50\varepsilon}H^2)$. The preceding bounds give $1 \ll D \ll x^\delta r$, so single density supplies $d \mid r$ with $x^{-\delta}D \le d \le D$. Write $r = dr_1$, $d \asymp \Delta$ and $\Delta_1 = x^{-5\varepsilon}\Delta$ as in [19, Lem. 4], and put

$$g = (v_1, v_2), \qquad g_0 = (q_0, \ell), \qquad \mathcal{K} = \max\{1, x^{5\varepsilon}H/V\}, \qquad m = r_1 q_0 u_1 [v_1, v_2] q_2.$$

The original products $rq_0u_1v_iq_2$ are squarefree, so m is squarefree and $(d, m) = 1$. Moreover,

$$\begin{aligned} \mathcal{K} \ll q_0, \qquad H \ll H_0/q_0, \qquad H_0 = x^{4\omega+\delta+7\varepsilon}, \\ \frac{R_0 Q^2 H}{q_0 g \mathcal{K} \Delta_1} \ll m \ll \frac{x^\delta R_0 Q^2 H}{q_0 g \Delta_1}. \end{aligned} \tag{A.12}$$

The moment estimate [19, Appendix, Lem. 27] uses $UV \asymp Q/q_0$. Its progression remainder is $O(N/q_0)$ because $q_0 \ll H_0$ and (A.9) give $N/q_0 \gg x^\eta$ for some fixed $\eta > 0$. In [19, Lems. 4–8], the removed diagonal contributes $\ll x^{o(1)}\Delta HVN \ll q_0 x^{-5\varepsilon+o(1)}\Delta NV^2$. Before differencing, separate the four Fourier weights by Taylor expansion on $d = d_0 + O(\Delta_1)$. Indeed, for $|h| \ll H$,

$$\Delta_1^j |\partial_d^j \varphi_H^\star(h)| \ll_j (\log x)^{O_j(1)} x^{-4j\varepsilon},$$

and likewise for $\varphi_H^{\star\star}$. A fixed finite family of common smooth cutoffs therefore suffices, with arbitrarily small power error. The fibers of $(h_1, h_2) \mapsto h_1(v_2/g) - h_2(v_1/g)$ have size $O(1 + gH/V) = O(g\mathcal{K})$. Thus Lemma 6's squared bound loses $\mathcal{K}^2$. For $s_2 \mid m$, fixing $h_1 \bmod (s_2, v_1/g)$ bounds the number of image values divisible by s_2, and likewise their differences, by $O(H^2/s_2 + H)$. Lemma 7's remainder is still $O(x^{-47\varepsilon}N^2)$. Lemma 6's m-supported divisor partition uses $O(\log x)$ prime divisors and has cost $x^{o(1)}$ for our polynomial-size m.

Accordingly, the target in [19, Lem. 8] becomes

$$\Sigma_5 \ll_\varepsilon \frac{s_2 q_0^2 g_0^2 N^2}{\mathcal{K}^2 x^{27\varepsilon}\mathcal{R}}, \qquad s_2 = (w_2, m), \qquad \mathcal{R} = \max\{x^{\delta+10\varepsilon}H^3, H^4\}. \tag{A.13}$$

Use its transformed parameters $w_1, w_2, z_1, \lambda, \widetilde{\lambda}, \Lambda, A, B$ and substitutions $d_{\mathrm{old}} = z_1 d$, $y = w_1\lambda$, $\widetilde{y} = w_1\widetilde{\lambda}$, retaining the unit conditions. The error is admissible since $\mathcal{K}^2 \ll q_0^2 g_0^2$.

Put $s = z_1/w_1$, $J_k = sk + (\lambda - \widetilde{\lambda})B$. Lemma 9's linearization is the integer identity $\widetilde{n} = (\widetilde{\lambda}n + skd)/\lambda$; it forces $w_2 \mid k$. After cancellation of w_2, both $\lambda/w_2, \widetilde{\lambda}/w_2$ are units modulo m. The class indicators cost at most $q_0 g_0$ choices, so $\Sigma_5 \ll_\varepsilon q_0 g_0 \sum_k \sup \Sigma_6(k)$ up to negligible Taylor errors, where

$$|k| \ll \frac{w_1|\Lambda|N}{x^{5\varepsilon}\Delta_1}, \qquad w_2 \mid k, \qquad (J_k, m) \le T, \qquad T = \max\{s_2^{-1}, H^{-1}\}\frac{x^{\delta+100\varepsilon}H^2N}{g\Delta_1}.$$

All divisions by λ and w_2 here are integer divisions.

Write $\lambda^* = (\lambda, \widetilde{\lambda})$, $u = \lambda/\lambda^*$, $v = \widetilde{\lambda}/\lambda^*$; negating $\lambda, \widetilde{\lambda}, k, A$ together makes $u, v > 0$. The sum vanishes unless $\lambda^* \mid sk$. Otherwise choose $0 \le F < u$ and $G \in \mathbb{Z}$ with $uG - vF = sk/\lambda^*$, and put $n = un_1 + Fd$, $\widetilde{n} = vn_1 + Gd$. The numbers $u, v, \lambda^*/w_2$ are units modulo m, and

$$\frac{G+B}{v} - \frac{F+B}{u} = \frac{J_k}{\lambda^* uv} \equiv \frac{J_k/w_2}{(\lambda^*/w_2)uv} \pmod{m}.$$

The homogeneous Möbius–CRT partition of [19, Lems. 10–11], at scale

$$\Delta^* = \min\{N/(|\Lambda|x^{5\varepsilon}), \Delta_1\},$$

gives the phase in Lemma A.7, with numerator $w_2 A_1 L_1$. Here A_1, A_2 are units and $L_1 = A_2(J_k/w_2) \bmod m$. Only the weights are translated. The partition costs $x^{2\varepsilon}(\Delta_1/\Delta^*)q_3$, where $q_3 \mid m/q_0$, and imposes modulus $q_0 q_3$. Equation (A.8) cancels q_3 at every rescaled length. Since $(w_2 A_1 L_1, m) = (J_k, m)$, it gives

$$\Sigma_6(k) \ll_\varepsilon \frac{x^{3\varepsilon}(J_k, m)}{q_0}\frac{\Delta_1}{\Delta^*}\left(m + N + \Delta^* + \frac{N\Delta^*}{m}\right). \tag{A.14}$$

The gcd sum [19, Lems. 15–16] gives $\sum_k (J_k, m) \ll_\varepsilon x^\varepsilon s_2 T$: apply it after $k = w_2 j$, using $(sw_2, m) = s_2$ and $T \gg x^{100\varepsilon} w_1|\Lambda|N/(x^{5\varepsilon}w_2\Delta_1)$. If $T < 1$, the admissible set is empty.

Put $\mathcal{W} = H^2\mathcal{R}$. Since $T \le x^{\delta+100\varepsilon}H^2N/(g\Delta_1)$ and $\Delta^* \le N$, comparison with (A.13) reduces to bounding

$$(\mathcal{A}, \mathcal{B}, \mathcal{C}) = \frac{\mathcal{K}^2 x^{\delta+131\varepsilon}\mathcal{W}}{gq_0^2 g_0}\left(\frac{1}{m}, \frac{1}{\Delta^*}, \frac{m}{N\Delta^*}\right). \tag{A.15}$$

The ranges in Lemma 8 and (A.12) give

$$\Delta_1, \Delta^* \gg \frac{N}{x^{\delta+55\varepsilon}H^2}, \qquad \frac{x^{54\varepsilon}MH^4}{gN\mathcal{K}} \ll m \ll \frac{x^{\delta-\varepsilon}MH^2}{g\Delta_1};$$

the last upper bound uses $R_0 Q^2 = x^{-\varepsilon}q_0 MH$. In particular, $\mathcal{K}^3 H/q_0^2 \ll H_0$ and $\mathcal{K}^3 H^2/q_0^2 \ll H_0^2$. Substitution, with $MN \asymp x$, now gives

$$\mathcal{A} \ll x^{-1+2\gamma+8\omega+3\delta+100\varepsilon}, \quad \mathcal{B} \ll x^{32\omega+10\delta-\gamma+400\varepsilon}, \quad \mathcal{C} \ll x^{1+48\omega+16\delta-4\gamma+400\varepsilon}.$$

Equation (A.9) bounds all three ratios, proving (A.13).

For larger γ, use density nesting and [15, Thm. 2.8(ii),(iv)], with Proposition 2.15 and Lemmas A.2 and A.3. This range begins at $1/4 + 14\omega + 4\delta$ and overlaps the preceding one by $(1 - 72\omega - 24\delta)/4 > 0$. Also $1 - 68\omega - 14\delta > 0$; the distances from $1/2 - \sigma$ to the two lower endpoints are $(1 - 48\omega - 16\delta - 4\sigma)/4$ and $(1 - 64\omega - 20\delta - 2\sigma)/2$. Small enough ε preserves all these margins. The residue classes restrict the fixed primitive class modulo P_I; all estimates are uniform in that class, I, and the coefficient bounds. $\square$

A.5. **A smooth factor on singly densely divisible moduli.**

Proof of the first assertion of Lemma 2.20. Put $\gamma := 1/2 - \sigma > 1/4 + 7\omega + 2\delta$. Apply [2, Lem. 3, (2.3)–(2.4)], using Proposition 2.15 below $x^{1/2-\varepsilon}$ and Lemmas A.1 and A.2 for the exceptional and zero-frequency terms. The parameters satisfy

$$x^{-\delta-O(\varepsilon)}N \ll R \ll x^{-2\varepsilon}N, \qquad RQ \ll x^{1/2+2\omega+O(\varepsilon)}, \qquad H := \frac{x^{\varepsilon}RQ^2}{q_0M}.$$

There is nothing to sum if $H < 1$. The complete moduli rq_0q_1, rq_0q_2 are singly dense, hence so is their least common multiple [15, Lem. 2.10(ii)]. Retaining the source's CRT unit restrictions, apply Lemma A.3 with $(d, d_1, d_2) = (q_0, rq_0q_1, q_2)$. Indeed, $N/q_0 \gg N/Q \gg x^{2\gamma-1/2-2\omega-\delta-O(\varepsilon)} > x^{12\omega+3\delta-O(\varepsilon)}$ and $N \leq [rq_0q_1, q_2]^3$. Footnote 1 supplies the corrected mean-factor average. For $g_0 := (q_0, \ell)$ the resulting estimate is

$$\frac{\Upsilon_{\ell,r}}{g_0Q^2N/q_0^2} \ll x^{\varepsilon+o(1)}\left\{q_0^{-5/3}\frac{(RQ^2)^{7/6}x^{\delta/6}}{MN^{1/2}} + \frac{1}{M}\right\}.$$

Substitute $RQ^2 \ll x^{1+4\omega+\delta+O(\varepsilon)}/N$ and $MN \asymp x$. The two terms in braces are respectively

$$O\big(x^{(1+28\omega+8\delta-4\gamma+O(\varepsilon))/6}\big) \quad \text{and} \quad O\big(x^{-1/2+o(1)}\big).$$

Choosing ε sufficiently small gives the required $x^{-\varepsilon}$ saving. ∎

A.6. **Localized coefficients and the minorant.** Write $(a, b, c, \xi) := (a_*, b_*, c_*, \xi_*)$ and $\zeta := 1 - c - a$. Use the six sifted functions and named-prime partitions of [19, Lem. 19], with footnote 3. Choose $0 < \tau \leq 10^{-10}$ after the strict margins in Proposition 2.22.

Lemma A.8 (Minorant coefficients)**.** *The localized coefficients below, their positive box majorants, the elementary Heath–Brown factors and their bounded convolutions satisfy* (2.4) *for* $x^{\varepsilon} \leq N \leq x^{C}$, *with one divisor exponent uniformly in endpoints, auxiliary coprimality and cutoffs. The smooth Type I and Type II inputs of Proposition 2.22 therefore give the asserted averaged distribution for the six sifted functions.*

Proof. Put $H := x^{a}$, $z := x^{\xi}$ and $M_0 := x^{1-c}$. For each partition, let r, s be its named-prime products, with $r < H$, $s < M_0/H$. Insert $\psi(v, z) = \sum_{d|v,\, P^+(d)<z} \mu(d)$. Terms with $rsd \leq M_0$ have a free factor of length at least x^{c} and give Type I after smooth subdivision. Otherwise order the primes of d_0 decreasingly and stop at the first product h with $rh \geq H$. Writing $d_0 = hd$ gives uniquely

$$rh/P^-(h) < H \leq rh < Hz = x^{b}, \qquad P^+(d) < P^-(h), \qquad rshd > M_0.$$

The variables rh and sdk in $n = rshdk$ have lengths between $x^{a+o(1)}$ and $x^{b+o(1)}$. This is [3, proof of Lem. 14, (4.5)–(4.8)], retaining the named-prime restrictions until separation.

For Siegel–Walfisz, put $L := \log x$, $Y := e^{\sqrt{L}}$. Fix $j \geq 2$ so all coefficients and positive majorants are bounded by $\tau_j(n)L^{C_0}$, independently of the saving and mesh. Rankin's bound and divisor moments give

$$\sum_{\substack{n \ll N \\ P^+(n) \leq Y}} \tau_j(n) \ll Ne^{-\log N/\log Y}L^{O_j(1)}, \qquad \sum_{\substack{n \ll N \\ p^2 | n \text{ for some } p > Y}} \tau_j(n) \ll NY^{-1}L^{O_j(1)}.$$

Both are $o(NL^{-A})$ for every A. Write each remaining integer as $n = pm$, where $p > Y$ is its unique largest prime. Fix its containing variable and all other factors. The permitted p form a bounded union of intervals: removing it from h either fixes $P^-(h)$ or leaves $h = p$ and $r < H \leq rp$. A Möbius factor becomes $-\mu$ of the cofactor; $|\mu|$ is unchanged. Handle integer, logarithmic and smooth weights by partial summation. For $q \leq L^{B}$, prime Siegel–Walfisz at endpoints $T \geq Y$ applies since $q \leq (\log T)^{2B}$; summing the remaining weights uses $\sum_{m \ll N} \tau(m)^{O(1)}/m \ll L^{O(1)}$. For auxiliary coprimality r_0, keep $(m, qr_0) = 1$ and remove primes dividing r_0, at cost $\ll \tau(r_0)NY^{-1}L^{O(1)}$. For $q > L^B$, fix a largest factor in a j-fold divisor representation:

$$\sum_{\substack{n \asymp N \\ n \equiv a_0 \ (q)}} \tau_j(n) \ll_j (N/q)L^{j-1} + N^{1-1/j}L^{j-2}.$$

The other factors have product $O(N^{1-1/j})$, including the isolated point in this bound. Use $q/\varphi(q) \leq \tau(q)$ and choose B after A to obtain (2.4) with a fixed divisor exponent. Convolution closure [15, proof of Lem. 3.4(iii)] uses $\|\alpha\|_1 \ll ML^{O(1)}$ and preserves that exponent.

Separate the restrictions into ratio-$1+h$ boxes, with $h = L^{-D}$ and singleton prime bins up to $Z = L^E$. Interior boxes give the Type I/II coefficients above; the positive full-box boundary majorant g satisfies, for C_1 independent of D, E,

$$\|g\|_1 \ll xL^{C_1}(hL + Z^{-1} + M_0^{-1/j}).$$

For a prime comparison this follows by summing $xL^{O(1)} \sum_{Z<\ell\ll\sqrt{x}} \ell^{-1} \sum_{p=\ell(1+O(h))} p^{-1}$ over integers; for $rshd = M_0(1 + O(h))$ use the preceding divisor count on a short interval. Named-prime boundaries obey the same bounds; $r < H$, $s < M_0/H$ remain exact. For $|e| \leq g$,

$$|\Delta(e; a_0 \bmod q)| \leq |\Delta(g; a_0 \bmod q)| + 2\|g\|_1/\varphi(q).$$

Choose D, E after the desired saving, then the Type I/II saving to absorb $L^{O_{D,E}(1)}$ boxes, using $\sum_{q\leq Q} \varphi(q)^{-1} \ll \log(2Q)$. Lemma A.1 handles total-product endpoints for $Q \leq x^{0.53}$. Every partition is independent of the modulus and primitive class. □

Lemma A.9 (Buchstab decomposition of the minorant)**.** *Suppose the Type I, II and III inputs used in Proposition 2.22 hold on a squarefree modulus family of level $Q \leq x^{0.53}$. Then $P - \mathfrak{b}_1 - \mathfrak{b}_2$ has the same averaged distribution estimate.*

Proof. Put $L = \log x$ and write $f \equiv g$ modulo the asserted discrepancy bound. For the three unsigned sums T_3, T_5, T_4 in the last display of [19, §4.4], Buchstab's identity and the preceding lemma give $P \equiv T_3 + T_5 - T_4$. Central indexed subset sums and initial semiprimes up to $\sqrt{3x}$ use the widened Type II range; prime squares cost $O((\sqrt{x} + Q)L^{O(1)})$ by counting square roots in primitive classes modulo squarefree q.

For T_3, apply [19, proof of Lem. 20, properties (1)–(4)] with τ in place of 10^{-10} and localization errors $O(\tau/1000)$. The preceding lemma supplies Siegel–Walfisz; the two grouping gaps are $1 - c - 3a/2 = 0.054225$ and $\zeta - (1 - 4\xi) = 0.01815$. Type I/II/III therefore give $T_3 \equiv 0$. Replacing a prime at scale N by $\Lambda/\log$ costs at most $x^{o(1)}(xN^{-1/2} + QN^{1/2})$, a power saving for $x^{\zeta-o(1)} \leq N \leq x^{a+o(1)}$.

The split in [19, proof of Lem. 21] gives $T_5 \equiv \mathfrak{b}_1$; the fifth-factor check is the argument in Proposition 2.21. For T_4, apply [19, proof of Lem. 22] with the complementary cutoff and equal-prime reversal in footnote 2. This gives $T_4 \equiv -U_3$, where U_3 is its five-prime sum Υ_3. Interchanging p_2, p_4 makes $\mathfrak{b}_2$ a subfamily of U_3; the footnote's pair/triple criterion gives a central indexed subset sum on the complement, except where $p_3 = p_4$. For $k = 4, 5$, let $D_{k,P}$ count configurations with a repeated prime $p \asymp P$. Counting the cofactor, including the isolated point, gives

$$\sum_{q\leq Q} |\Delta(D_{k,P}; a_q \bmod q)| \ll x^{o(1)}(x/P + QP), \qquad x^{\xi} \ll P \ll x^{(1-(k-2)\xi)/2},$$

This is a fixed power saving for $Q \leq x^{0.53}$, so $U_3 \equiv \mathfrak{b}_2$ and the result follows. □

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