ON THE RATIO OF $R(k,\ell)$ AND $R(k,\ell+1)$

Abstract. For fixed $k \geq 2$, we prove that $\lim_{\ell\to\infty} \frac{R(k,\ell+1)}{R(k,\ell)} = 1$. This answers a question of Erdős and is based on dependent random choice. The proof is due to an internal model at OpenAI.

1. Background

Let $R(k,\ell)$ denote the standard off-diagonal Ramsey number, that is, the smallest number N such that every N-vertex graph contains a K_k or an independent set of order ℓ. We use the convention $R(1,\ell) = 1$. Answering a question of Erdős [3, p. 99], we prove the following result.

Theorem 1. *For every fixed integer $k \geq 2$,*

$$\lim_{\ell\to\infty} \frac{R(k,\ell+1)}{R(k,\ell)} = 1.$$

Remark 1. The proof gives a quantitative bound. For each fixed $k \geq 2$, there is a constant $c_k > 0$ such that

$$\frac{R(k,\ell+1)}{R(k,\ell)} \leq 1 + \ell^{-c_k}$$

for all sufficiently large ℓ. We do not attempt to optimize c_k.

Ramsey numbers are a classical topic in combinatorics and have seen several recent breakthroughs. These include the exponential improvement to the upper bound for diagonal Ramsey numbers by Campos, Griffiths, Morris and Sahasrabudhe [2] (see also [6]) and a proof by Mattheus and Verstraete [7] that implies that $R(4,\ell) = \ell^{3+o(1)}$. Despite this progress, our understanding of off-diagonal Ramsey numbers remains limited. In general, the best lower bounds for $R(k,\ell)$ come from the work of Bohman and Keevash [1], which gives

$$R(k,\ell) \gg_k \ell^{(k+1)/2} (\log \ell)^{1/(k-2)-(k+1)/2},$$

whereas the classical Erdős–Szekeres upper bound [4], giving $R(k,\ell) \ll_k \ell^{k-1}$, has only been improved by logarithmic factors. Nevertheless, Theorem 1 shows that the ratio of consecutive terms tends to 1 by considering the structure of a critical graph for $R(k,\ell+1)$ and applying dependent random choice. The power of dependent random choice in this setting comes from the following observation: if $R(k,\ell+1) \geq (1+\varepsilon)R(k,\ell)$, then a critical graph for $R(k,\ell+1)$ has minimum degree at least roughly $\varepsilon R(k,\ell+1)$. The fact that such a hypothetical critical graph is dense is contrary to the expectation that the optimal graphs should be sparse, and this tension drives the proof below.

2. Proof

The proof of Theorem 1 requires three external inputs. The first is the Erdős–Szekeres bound on Ramsey numbers [4].

Lemma 1 (Erdős–Szekeres)**.** *For all $k, \ell \geq 2$,*

$$R(k,\ell) \leq \binom{k+\ell-2}{k-1}.$$

The second is the off-diagonal lower bound obtained via a standard application of the probabilistic method.

Lemma 2. *For every fixed $k \geq 3$, as $\ell \to \infty$,*

$$R(k, \ell) \gg_k (\ell / \log \ell)^{k/2}.$$

Finally, we require dependent random choice; see Fox–Sudakov [5, Lemma 2.1], or [8, Theorem 1.7.5].

Lemma 3 (Dependent random choice)**.** *Let G be an N-vertex graph of average degree d. Let q, s, m be positive integers. Then there is a set $U \subseteq V(G)$ such that every s-subset of U has at least m common neighbours and*

$$|U| \geq \frac{d^q}{N^{q-1}} - \binom{N}{s} \left(\frac{m-1}{N} \right)^q .$$

Proof of Theorem 1. The case $k = 2$ is immediate, since $R(2, \ell) = \ell$. Fix $k \geq 3$, and let

$$s = \left\lceil \frac{k}{2} \right\rceil, \qquad t = \left\lfloor \frac{k}{2} \right\rfloor, \qquad q = k^2.$$

Choose a graph G on $R(k, \ell+1) - 1$ vertices with no K_k and with $\alpha(G) \leq \ell$. We first note that

$$\delta(G) \geq R(k, \ell+1) - R(k, \ell) - 1. \tag{1}$$

Indeed, for $v \in V(G)$, the graph induced on $V(G) \setminus (N(v) \cup \{v\})$ has no K_k and no independent set of order ℓ, since such an independent set together with v would have order $\ell + 1$. Hence

$$|V(G) \setminus (N(v) \cup \{v\})| \leq R(k, \ell) - 1.$$

We now apply Lemma 3 to G with $m = R(t, \ell+1)$, and write $N = R(k, \ell+1) - 1$. Since the minimum degree of G is at least $R(k, \ell+1) - R(k, \ell) - 1$, there is a set $U \subseteq V(G)$ such that every s-subset of U has at least $R(t, \ell+1)$ common neighbours and

$$|U| \geq \frac{(R(k, \ell+1) - R(k, \ell) - 1)^q}{N^{q-1}} - \binom{N}{s} \left(\frac{R(t, \ell+1) - 1}{N} \right)^q \tag{2}$$

for our chosen value of q. We next claim that $|U| \leq R(s, \ell+1) - 1$. The graph $G[U]$ has no independent set of order $\ell + 1$. If $G[U]$ contained a K_s, then its common neighbourhood would have at least $R(t, \ell+1)$ vertices. This common neighbourhood contains no independent set of order $\ell + 1$, and therefore contains a K_t. Together with the K_s, this gives a K_k, a contradiction. Thus $G[U]$ has no K_s and no independent set of order $\ell + 1$, proving the claim.

Combining this claim with (2), we obtain

$$\left(\frac{R(k, \ell+1) - R(k, \ell) - 1}{N} \right)^q \leq \frac{R(s, \ell+1) - 1}{N} + \binom{N}{s} \frac{(R(t, \ell+1) - 1)^q}{N^{q+1}}. \tag{3}$$

By Lemma 1,

$$R(s, \ell+1) \ll_k \ell^{s-1}, \qquad R(t, \ell+1) \ll_k \ell^{t-1},$$

where the second estimate is also valid when $t = 1$, by the convention $R(1, \ell+1) = 1$. By Lemma 2,

$$N = R(k, \ell+1) - 1 \gg_k \left(\frac{\ell}{\log \ell} \right)^{k/2} = \ell^{k/2 - o(1)}.$$

The first term on the right-hand side of (3) is therefore $o(1)$. For the second term, using $\binom{N}{s} \leq N^s / s!$, we get

$$\binom{N}{s} \frac{(R(t, \ell+1) - 1)^q}{N^{q+1}} \ll_{k,q} \frac{\ell^{q(t-1)}}{N^{q-s+1}} = o(1),$$

where the last step follows from $N \gg_k \ell^{k/2 - o(1)}$, the balanced choice of s, t, and $q = k^2$. Thus the right-hand side of (3) is $o(1)$. Taking q-th roots gives

$$\frac{R(k, \ell+1) - R(k, \ell)}{R(k, \ell+1) - 1} \to 0,$$

and the desired ratio estimate follows. ☐

References

[1] T. Bohman and P. Keevash, *The early evolution of the H-free process*, Invent. Math. **181** (2010), 291–336.

[2] M. Campos, S. Griffiths, R. Morris, and J. Sahasrabudhe, *An exponential improvement for diagonal Ramsey*, 2025, Ann. of Math. (2), to appear; arXiv:2303.09521.

[3] P. Erdős, *Some unsolved problems in graph theory and combinatorial analysis*, Combinatorial Mathematics and its Applications, Academic Press, London, 1971, Proc. Conf., Oxford, 1969, pp. 97–109.

[4] P. Erdős and G. Szekeres, *A combinatorial problem in geometry*, Compositio Math. **2** (1935), 463–470.

[5] J. Fox and B. Sudakov, *Dependent random choice*, Random Structures Algorithms **38** (2011), 68–99.

[6] P. Gupta, N. Ndiaye, S. Norin, and L. Wei, *Optimizing the CGMS upper bound on Ramsey numbers*, 2024, arXiv:2407.19026.

[7] S. Mattheus and J. Verstraete, *The asymptotics of $r(4,t)$*, Ann. of Math. (2) **199** (2024), 919–941.

[8] Y. Zhao, *Graph theory and additive combinatorics: Exploring structure and randomness*, Cambridge University Press, 2023.