

Read and understand the following paper. The key manipulations occur in Appendices A and B, make sure you understand those to the maximum precision

```
\documentclass[amsmath,amssymb,aps,nofootinbib,prd,
reprint,superscriptaddress]{revtex4-2}
```

```
\usepackage[T1]{fontenc}
\usepackage[dvipsnames]{xcolor}
\usepackage{hyperref}
\hypersetup{colorlinks=true,citecolor=RoyalBlue,linkcolor
=RoyalBlue,urlcolor=RoyalBlue}
```

```
\let\originalleft\left
\let\originalright\right
\renewcommand{\left}
{\mathopen{}\mathclose\bgroup\originalleft}
\renewcommand{\right}{\aftergroup\egroup\originalright}
\mathcode \*="8000
{\catcode \*=\active\gdef*
{\mathclose{} \, \mathopen{}}}
```

```
\newcommand{\ab}[1]{\left|#1\right)}
\newcommand{\av}[1]{\left|\langle#1\rangle\right)}
\newcommand{\br}[1]{\left[#1\right)}
\newcommand{\cu}[1]{\left\{#1\right\}}
\newcommand{\pa}[1]{\left(#1\right)}
\newcommand{\ed}{\mathop{\!|\!}\mathrm{d}}
\newcommand{\pd}{\mathop{\!|\!}\mathrm{partial}}
\newcommand{\SD}{\mathcal{R}_1}
\newcommand{\sg}{\operatorname{sg}}
\newcommand{\tlam}{\tilde{\lambda}}
```

```
\begin{document}
```

```
\title{Single-minus gluon tree amplitudes are nonzero}
```

```
\author{
Alfredo Guevara,\textsuperscript{1}
Alexandru Lupsasca,\textsuperscript{2,3}
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Andrew Strominger,\textsuperscript{5}
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on behalf of OpenAI\\  
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\begin{abstract}  
 Single-minus tree-level  $n$ -gluon scattering amplitudes  
 are reconsidered.  
 Often presumed to vanish, they are shown here to be  
 nonvanishing for certain 'half-collinear'  
 configurations existing in Klein space or for  
 complexified momenta.  
 We derive a piecewise-constant closed-form expression  
 for the decay of a single minus-helicity gluon into  $n-1$   
 plus-helicity gluons as a function of their momenta.  
 This formula nontrivially satisfies multiple consistency  
 conditions including Weinberg's soft theorem.  
 \end{abstract}

\maketitle

The laws of physics are succinctly encoded in  
 $\text{\emph{scattering amplitudes}}$ , which give the quantum  
 probabilities for any given collection of incoming particles  
 to collide and produce any given collection of outgoing  
 particles.  
 These amplitudes may be systematically derived from  
 the Feynman diagram expansion, which perturbatively  
 sums over all possible quantum processes.  
 Theoretical results from the Feynman diagram expansion  
 of the Standard Model agree with experiment to an  
 unprecedented 14 decimal places  
 $\text{\cite{Aoyama:2017uqe,Morel:2020dww,Fan:2022eto}}$ .

In practice, the computation of scattering amplitudes can  
 be extremely difficult.\footnote{The aforementioned  
 agreement between theory and experiment required over  
 a half century of analytic and numerical work.}  
 Among other obstacles, the growth in the number of  
 Feynman diagrams for an  $n$ -particle amplitude is faster  
 than exponential in  $n$ .  
 However, despite this apparent complexity, cancellations

lead in a variety of contexts to a very simple final answer. This indicates that our present understanding of the quantum laws of physics is seriously incomplete and that a more efficient formulation is needed.

The last few decades have seen much effort in this direction and delivered promising insights; see, [\emph{e.g.}, \cite{Dixon:1996wi,Witten:2003nn,Roiban:2004yf,Bern:2005iz,Britto:2005fq,Arkani-Hamed:2012zlh,Arkani-Hamed:2013jha}](#).

A prominent example of this phenomenon arises in the tree-level color-ordered scattering of gluons---the particles that mediate the strong force and comprise Yang--Mills theory.

Naively, the  $n$ -gluon scattering amplitude involves order  $n!$  terms.

Famously, for the special case of MHV (maximally helicity violating) tree amplitudes, Parke and Taylor [\cite{Parke:1986gb}](#) gave a simple and beautiful, closed-form, single-term expression for all  $n$ .

By definition,  $n$ -gluon MHV amplitudes have  $2$  minus-helicity particles and  $n-2$  plus-helicity gluons, which for generic (complexified) kinematics at tree level is the maximally allowed number [\cite{Parke:1986gb,Grisaru:1977px,Elvang:2013cua,Dixon:1996wi,Bern:1994zx}](#).

This gives them a privileged role in the theory, enabling their use as efficient building blocks for the full Yang--Mills theory.

In general,  $n-2$  is actually [\emph{not}](#) the maximally allowed number of plus gluons.

In this paper, we show that  $n-1$  plus (or 'single-minus') amplitudes are in fact allowed [\footnote{Witten~\cite{Witten:2003nn}}](#) notes that single-minus amplitudes are supported at a point in twistor space; see also [\cite{Roiban:2004yf}](#). with restricted 'half-collinear' kinematics. [\footnote{The half-collinear condition can be viewed as restricting the ingoing and outgoing momenta to a one-dimensional null circle on the celestial torus at the boundary of Klein space.}](#)

The amplitude is divided into chambers whose walls are

regions where sums of various subsets of the half-collinear momenta are orthogonal as described below. The (stripped) amplitudes are piecewise-constant integers in each chamber.

The assignment of its value to each chamber is determined from the perturbative Berends--Giele recursion \cite{Berends:1987me}, which is equivalent to Feynman diagrams.

Moreover, for the special kinematic region corresponding to single-minus decay into  $n-1$  plus, we give a simple formula for all  $n$ .

In this special region, the stripped amplitude only takes the values of  $+1$ ,  $-1$ , or  $0$ .

The key formula \eqref{eq:Simplification} for the amplitude in this region was first conjectured by GPT-5.2 Pro and then proved by a new internal OpenAI model. The solution was checked by hand using the Berends--Giele recursion and was moreover shown to nontrivially obey the soft theorem, cyclicity, Kleiss--Kuijf, and  $\mathsf{U}(1)$  decoupling identities---none of which are evident from direct inspection.

The structural role of these single-minus amplitudes in Yang--Mills theory remains to be understood.

We note that, while our expression is a dramatic simplification of the direct Feynman-diagram expression, it is entirely possible that a yet simpler expression may be obtained with a clever choice of analytic continuation, variables or basis, even outside the single-minus decay channel.

We suspect that there are more interesting insights to come with our methodology and hope that this paper is a step on the road to a more complete understanding of the inner structure of scattering amplitudes.

Single-minus amplitudes also arise in self-dual Yang--Mills theory (SDYM) \cite{Cangemi:1996rx}, a restricted sector of Yang--Mills, and potentially resolve a puzzle therein.

In general, the tree amplitudes of the Feynman expansion are thought to be equivalent to the fully nonlinear classical theory.

However, on the one hand the classical solution space of SDYM is extremely nontrivial

\cite{WardWells:1990twistor,Ward:1977sdym\_twistor,ADHM:1978instantons}, while the tree diagrams were previously supposed to yield trivial two-point and three-point expressions.

The latter seem insufficient to reproduce the former. Potentially, the single-minus tree amplitudes in SDYM found here resolve this tension.

This paper is organized as follows.

In Sec.~\ref{sec:Recursion}, we set up notation, describe the standard MHV amplitudes, explain how half-collinear single-minus amplitudes evade the usual no-go condition, and then derive the general Berends--Giele recursion relation.

The solution passes various consistency checks, including the soft theorem, and we provide explicit formulas up to  $n=6$  points, where there are already 32 terms.

In Sec.~\ref{sec:Formula}, we restrict to a special kinematic channel denoted  $\mathcal{SD}$  with one ingoing minus and  $n-1$  outgoing plus gluons.

There, using various identities through  $n=6$ , we find the answer can be expressed as a signed product of  $n-2$  projection operators.

This motivates a guess for the all- $n$  formula, which we verify directly via the Berends--Giele recursion.

We derive a multi- $\delta$ -function identity in App.~\ref{app:Master} and give more details of the single-minus specialization of the Berends--Giele recursion in App.~\ref{app:Derivation}.

Further details of our analysis, including a longer general formula for the single-minus amplitude outside of  $\mathcal{SD}$ , will appear elsewhere.

Our main result immediately leads to a number of extensions.

The construction generalizes directly from gluon to graviton amplitudes and has a simple supersymmetrization.

The results should transform under the  $\mathcal{SS}$ -algebra, the  $\mathcal{L}_{1+\infty}$  algebra

\cite{Guevara:2021abz,Strominger:2021mtt}, and their supersymmetric extensions.

In the context of celestial holography, the Mellin transform of the amplitudes in some sectors is given by Lauricella functions.

These results will be reported elsewhere.

\subsection{Notation and useful identities}

This subsection defines our notation\footnote{Our conventions are close to those of \cite{Witten:2003nn}, except for a factor of 2 on the LHS of (2.7) therein.} and presents several useful identities.

We use spinor-helicity variables for massless momenta

\cite{Elvang:2013cua}

\begin{align}

$p_{\{\alpha\dot{\alpha}\}}=\lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}}$ ,

where  $(\lambda,\tilde{\lambda})$  are \emph{real} spinors in  $(2,2)$  Klein signature.

As usual in the description of scattering, it is convenient to fix a suitable Lorentz and helicity frame to perform the calculation, which can be restored at the end.

With the benefit of hindsight, we write

\begin{align}

\label{eq:Conventions}

$|i\rangle=\lambda_i=\lambda_{\{1,z_i\}}$ , \quad

$|i]=\tilde{\lambda}_i=\omega_i\lambda_{\{1,\tilde{z}_i\}}$ ,

\end{align}

with  $z_i$  and  $\tilde{z}_i$  real and independent.

We use standard brackets for contracting helicity spinors,

\begin{align}

$\langle ij\rangle\langle ij\rangle=\lambda_i\lambda_j$

$=\epsilon_{\{\alpha\beta\}}\lambda_i^{\alpha}\lambda_j^{\beta}$ ,

\|

$[ij]=\tilde{\lambda}_i\tilde{\lambda}_j$

$=\epsilon_{\{\dot{\alpha}\dot{\beta}\}}\tilde{\lambda}_i^{\dot{\alpha}}\tilde{\lambda}_j^{\dot{\beta}}$ ,

\end{align}

with  $p_{\{ij\}}^2=(p_i+p_j)^2=\langle ij\rangle\langle ij\rangle+[ij][ij]$ .

In our parameterization,

\begin{align}

$\langle ij\rangle=z_{\{ij\}}$ , \quad

$[ij]=\omega_i\omega_j\tilde{z}_{\{ij\}}$ ,

\end{align}

where  $z_{\{ij\}}\equiv z_i-z_j$  and

$\tilde{z}_{\{ij\}}\equiv\tilde{z}_i-\tilde{z}_j$ .

We take our polarization vectors to be

$$\begin{aligned} & \text{\label{eq:PolarizationVectors}} \\ & \epsilon_{-j} = \sqrt{2} \frac{|r\rangle \langle j|}{[rj]}, \quad \epsilon_{+k} = \sqrt{2} \frac{|k\rangle \langle r|}{\langle av{rk}\rangle}, \end{aligned}$$

where  $|r\rangle$  and  $|r]$  are arbitrary reference spinors.

We note that with our conventions~\eqref{eq:Conventions} for fixing the little group frame,  $\epsilon^{\pm}$  has mass dimension  $\pm 1$ , which will affect the mass dimensions of the amplitudes given below.

To avoid proliferation of factors of  $2\pi$ , we normalize all  $\delta$ -functions such that

$$\begin{aligned} & \text{\label{eq:DeltaNormalization}} \\ & \int \delta(x) dx = 2\pi, \quad \frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon} \stackrel{\epsilon \rightarrow 0}{=} -i\delta(x). \end{aligned}$$

Throughout this paper, we use the standard Feynman propagator  $1/(p^2+i\epsilon)$ .

Other prescriptions have been considered in Klein signature~\cite{Arkani-Hamed:2009hub}.

To clarify our conventions, with our normalization, the  $n$ -point MHV (double-minus) color-ordered tree amplitude  $\mathcal{A}_n^{\text{MHV}}(1^+, \dots, r^-, \dots, s^-, \dots, n^+)$  is

$$\begin{aligned} & \text{\label{eq:NoEpsilonMHV}} \\ & \mathcal{A}_n^{\text{MHV}} = i \frac{\langle av{rs}\rangle^4}{\langle av{12}\rangle \langle av{23}\rangle \cdots \langle av{n1}\rangle}, \quad \delta^4(\sum_{k=1}^n p_k). \end{aligned}$$

In fact, for full generality, we will need to be careful about the  $\epsilon$  prescription.

We therefore introduce the regularized Parke-Taylor factor

$$\begin{aligned} & \text{\label{eq:PT}} \\ & \text{PT}_{\text{cyc}} = \prod_{k=1}^n \frac{[k, k+1]}{p_{k, k+1}^2 + i\epsilon} \\ & = \prod_{k=1}^n \frac{1}{z_{k, k+1} + i\epsilon} \text{sg}_{k, k+1}, \end{aligned}$$

with  $n+1$  equiv 1\$, and where we have defined

$$\begin{aligned} & \text{\begin{align}} \\ & \text{\sg}_{ij} = \text{\sg}\text{\pa}\{[\text{\tlam}_i\text{\tlam}_j]\} \\ & \text{\end{align}} \end{aligned}$$

in the frame  $\sim$  [\eqref{eq:Conventions}](#). [\footnote](#){Without fixing a frame, we would take

$\text{\sg}_{ij} = \text{\sg}\text{\pa}\{[ij]\text{\av}\{ir\}\text{\av}\{jr\}\}$ , where  $\text{\r}\text{\rangle}$  is any fixed reference spinor.

In  $\text{\mathcal}\{A\}_n^{\text{\rm MHV}}$ , the difference in choices of  $\text{\r}\text{\rangle}$  can be absorbed by  $\text{\epsilon}$ .

Here,  $\text{\sg}(x) = 2\Theta(x) - 1$  denotes the sign function and  $\Theta(x)$  is the step function.

The MHV tree amplitude can then be written using

$\text{\mathrm}\{PT\}_n^{\text{\rm cyc}}$  as

$$\begin{aligned} & \text{\label}\{eq:MHV\} \\ & \text{\mathcal}\{A\}_n^{\text{\rm MHV}} = i\text{\av}\{rs\}^4 \text{\mathrm}\{PT\}_n^{\text{\rm cyc}} \text{\delta}^4 \text{\pa}\{\sum_{k=1}^n p_k\}. \\ & \text{\end{align}} \end{aligned}$$

Away from walls where  $\text{\av}\{k, k+1\} = 0$ , we may ignore the  $\text{\epsilon}$  prescription and [\eqref{eq:MHV}](#) reduces to  $\sim$  [\eqref{eq:NoEpsilonMHV}](#).

It will also be useful to define an 'incomplete' or 'open chain' Parke-Taylor factor as

$$\begin{aligned} & \text{\label}\{eq:PTC\} \\ & \text{\mathrm}\{PT\}_{1 \cdots n} = \prod_{k=1}^{n-1} \frac{1}{[k, k+1] \{p^2_{k, k+1} + i\epsilon\}} \\ & = \prod_{k=1}^{n-1} \frac{1}{z_{k, k+1} + i\epsilon \text{\sg}_{k, k+1}}. \\ & \text{\end{align}} \end{aligned}$$

This incomplete factor  $\text{\mathrm}\{PT\}_{1 \cdots n}$  is what naturally appears in [App.  \$\sim\$  \ref{app:Derivation}](#) inside the Berends-Giele recursion as the denominator of the off-shell current with momentum  $p_{1 \cdots n}$ : the cyclic factor is 'opened' because one leg is off-shell.

## \section{Single-minus amplitudes}

### \label{sec:Recursion}

In this section, we first explain why the standard argument that the single-minus  $n$ -particle tree amplitudes vanish in fact fails when all the external



Hence, single-minus amplitudes vanish for generic kinematics.

The loophole in the argument is that we cannot choose  $|\mathbf{r}\rangle = |\mathbf{1}\rangle$  if  $\langle \mathbf{1} | \mathbf{a} \rangle = 0$  for any  $|\mathbf{a}\rangle$ , as the polarization vectors  $\epsilon^+_{\mathbf{a}}$  would become singular.

Therefore, we cannot conclude that the amplitude vanishes on the locus where  $\langle \mathbf{1} | \mathbf{a} \rangle = 0$ .

In fact, the single-minus 3-point amplitude (also anti-MHV) is known to have a factor  $\delta(\langle \mathbf{1} | \mathbf{2} \rangle) \delta(\langle \mathbf{1} | \mathbf{3} \rangle)$  restricting to this locus. Moreover, it can be shown by induction that the  $n$ -point amplitude may only be supported when  $\langle \mathbf{1} | \mathbf{ij} \rangle = 0$ .

To express the fact that it is supported in the half-collinear regime, the single-minus tree-level  $n$ -gluon amplitude  $\mathcal{A}_n(1^-, 2^+, \dots, n^+)$  can be written as

$$\begin{aligned} & \text{\label{eq:AnsatzP}} \\ & \mathcal{A}_n = i^{2-n} \frac{\langle \mathbf{1} | \mathbf{r}_1 \rangle^{n+1}}{\langle \mathbf{1} | \mathbf{r}_2 \rangle \langle \mathbf{1} | \mathbf{r}_3 \rangle \cdots \langle \mathbf{1} | \mathbf{r}_n \rangle} A_{1 \cdots n} \\ & \prod_{a=2}^n \delta(\langle \mathbf{1} | \mathbf{a} \rangle) \not\propto \\ & \quad \times \delta^2(\sum_{i=1}^n \langle \mathbf{1} | \mathbf{r}_i \rangle \mathbf{t}_{\mathbf{1} | \mathbf{a}_i}). \end{aligned}$$

Here, we introduced the stripped amplitude  $A_{1 \cdots n}$ .

The  $\delta$ -functions imposing  $\langle \mathbf{1} | \mathbf{a} \rangle = 0$  simply ensure the full amplitude is supported only in the half-collinear regime, while the  $\delta$ -functions in  $\mathbf{t}_{\mathbf{1} | \mathbf{a}}$  enforce the remaining components of momentum conservation.

The prefactor ensures that  $\mathcal{A}_n$  has the correct little-group scaling for a single-minus amplitude. The collinear  $\delta$ -functions ensure that this prefactor and the  $\mathbf{t}_{\mathbf{1} | \mathbf{a}}$   $\delta$ -functions are independent of the reference spinor  $|\mathbf{r}\rangle$ , as long as it is chosen so that  $\langle \mathbf{1} | \mathbf{r} \rangle \neq 0$ .

The interest is in the stripped amplitude  $A_{1 \cdots n}$ , which carries no helicity weight and depends only on kinematics.

In the frame  $\text{\ref{eq:Conventions}}$  and picking  $|\mathbf{r}\rangle = (0, 1)$ ,  $A_{1 \cdots n}$  is a function only of  $\langle \mathbf{1} | \mathbf{ij} \rangle$ .

$\{\tilde{\lambda}_i\}$ , and  $\text{\eqref{eq:AnsatzP}}$  becomes  $\text{\footnote{Some readers may find more intuitive an alternative expression, obtained in the frame } $\tilde{\lambda}_i \to \frac{1}{\sqrt{|\omega|}} \tilde{\lambda}_i$ ,  $\lambda \to \sqrt{|\omega|} \lambda$ , which renders the polarizations  $\text{\eqref{eq:Polarization}}$  dimensionless and gives  $\mathcal{A}_n$  scaling as  $\omega^{-n}$ :$

```
\begin{align*}
\mathcal{A}_n \to i^{2-n} \text{ab}\{\frac{\omega_1}{\omega_2 \omega_3 \dots \omega_n}\} A_{1 \dots n} \prod_{a=2}^n \delta(z_{1a}) \delta^2(p_{\sum_{i=1}^n \lambda_i}).
\end{align*}
\begin{align}
\text{\label{eq:Ansatz}}
\mathcal{A}_n = i^{2-n} A_{1 \dots n} \prod_{a=2}^n \delta(z_{1a}) \delta^2(p_{\sum_{i=1}^n \lambda_i}).
\end{align}
```

We will sometimes use the shorthand

```
\begin{align}
\delta_{1 \dots n} = i^{1-n} \prod_{k=1}^{n-1} \delta(z_{k,k+1})
\end{align}
```

to denote these half-collinear  $\delta$ -functions.

$\text{\subsection{The recursion relation}}$

The first main result of this paper is the recursion relation presented in  $\text{\eqref{eq:Recursion}}$  below.

This relation determines all  $n$ -particle single-minus tree amplitudes.

Solving it is equivalent to, but slightly simpler than, summing the Feynman diagrams for these amplitudes.

For any ordered list  $S = (q, \dots, p)$ , we first define the list momentum  $\lambda_S = \sum_{i \in S} \tilde{\lambda}_i$  using the frame  $\text{\eqref{eq:Conventions}}$ .  $\text{\footnote{In a generic frame, } $\lambda_S = \sum_{i \in S} \lambda_i \langle r |$  for any reference spinor  $|r\rangle$ .$

The dependence of both this expression and the sign functions on  $|r\rangle$  drops out on the support of the collinear  $\delta$ -functions.

We then define the  $\text{\emph{preamplitude}}$   $\bar{A}_S$  by taking

```

\begin{align}
\bar{A}_{[q]}=1,\quad
\bar{A}_{[qp]}=0,
\end{align}

```

when  $|S|=1$  and  $|S|=2$ , and extending recursively to  $|S|\geq 3$  via

```

\begin{align}
\label{eq:Preamplitudes}
\bar{A}_{[q\cdots p]}=-
\sum_{\{\text{o.p.}\}}V_{\{\tilde{\lambda}_{S_1}\}\cdots\{\tilde{\lambda}_{S_A}\}}\prod_{a=1}^A\bar{A}_{[S_a]},
\end{align}

```

where the sum is over all ordered partitions of  $(q\cdots p)=(S_1|S_2|\cdots|S_A)$  into  $A\geq 3$  parts.

Here, the vertex term  $V_{\{\tilde{\lambda}_{S_1}\}\cdots\{\tilde{\lambda}_{S_k}\}}$  is defined by

```

\begin{align}
\label{eq:Vertex}
V_{\{\tilde{\lambda}_1\}\cdots\{\tilde{\lambda}_n\}}&=\prod_{k=1}^{n-1}\text{sg}_{k,k+1}\Theta\left[\frac{\text{br}\{\tilde{\lambda}_1\}\cdots\text{br}\{\tilde{\lambda}_{k+1}\}\cdots\text{br}\{\tilde{\lambda}_n\}}{\text{br}\{\tilde{\lambda}_k\}\tilde{\lambda}_{k+1}}\right],
\end{align}

```

in the case where each block  $S_a$  contains only one element.

We also take  $V_{\{\tilde{\lambda}_1\}}=1$ .

Having determined  $\bar{A}_{[S]}$ , the stripped amplitude  $A_{[1\cdots n]}$  itself is then given by

```

\begin{align}
\label{eq:Recursion}
A_{[1\cdots n]}&=-
\sum_{\{\text{o.p.}\}}\widehat{\text{PT}}_{\{\tilde{\lambda}_{S_1}\}\cdots\{\tilde{\lambda}_{S_A}\}}\prod_{a=1}^A\bar{A}_{[S_a]},
\end{align}

```

where the ordered partition of  $(2\cdots n)$  now has  $A\geq 1$  parts, while

```

\widehat{\text{PT}}_{\{\tilde{\lambda}_1\}\cdots\{\tilde{\lambda}_n\}}=V_{\{\tilde{\lambda}_1\}\cdots\{\tilde{\lambda}_n\}}-\bar{V}_{\{\tilde{\lambda}_1\}\cdots\{\tilde{\lambda}_n\}},
\end{align}

```

where  $\bar{V}_{\{\tilde{\lambda}_1\}\cdots\{\tilde{\lambda}_n\}}$  is given by  $\text{eqref}\{\text{eq:Vertex}\}$  with a  $++$  sign within the argument of  $\Theta$ .

We may refer to this object as `\emph{on-shell Parke--`

Taylor}, as it is related to  $\mathrm{PT}_{1\cdots n}$  by a standard LSZ reduction procedure fleshed out in App.~\ref{app:Derivation}.

It is also useful to note that the incomplete Parke-Taylor factor  $\mathrm{PT}_{1\cdots n}$ , the collinear  $\delta$ -functions and the vertex function  $V_{\{1\cdots n\}}$  are related via the useful identity

$$\begin{aligned} & \label{eq:PT} \\ & \mathrm{PT}_{1\cdots n} \delta_{1\cdots n} \\ & V_{\{1\cdots n\}} \notag \\ & = \sum_{j=1}^{n-1} \frac{\mathrm{PT}_{1\cdots j} \mathrm{PT}_{j+1\cdots n}}{\rho_{1\cdots n}^2 + i\epsilon} \mathrm{PT}_{1\cdots j} \\ & \mathrm{PT}_{j+1\cdots n}, \end{aligned}$$

which follows from a master identity given in App.~\ref{app:Master}.

\subsection{Consistency checks}  
\label{sec:Properties}

It follows from the definition \eqref{eq:AnsatzP} that the stripped amplitudes  $A_{12\cdots n}$  satisfy the following properties:

\begin{enumerate}

\item Cyclicity:

\begin{aligned}

\label{eq:Cyclicity}

$$A_{12\cdots n} = A_{2\cdots n1}.$$

\end{aligned}

\item Reflection symmetry:

\begin{aligned}

$$A_{12\cdots n} = (-1)^n A_{n\cdots 21}.$$

\end{aligned}

\item  $\mathcal{U}(1)$  decoupling:

\begin{aligned}

$$A_{12\cdots n} + A_{13\cdots n2} + A_{14\cdots n23} + \dots = 0.$$

\end{aligned}

\item Kleiss--Kuijf (KK) relations.

For instance, at  $n=5$ ,

\begin{aligned}

$$A_{12345} + A_{12354} + A_{12435} + A_{14235} = 0.$$

```

\end{align}
(The general form of these relations can be found,
\emph{e.g.}, in~\cite{Elvang:2013cua}).
\item Weinberg's soft
theorem:\footref{SoftTheoremFootnote}
\begin{align}\label{eq:softm}
\lim_{\{\omega_n\to 0\}}A_{\{1\cdots n\}}=\frac{1}{
\{2\}\pa{\sg_{\{n-1,n\}}+\sg_{\{n1\}}A_{\{1\cdots n-1\}},.
\end{align}
\end{enumerate}

```

It is far from evident that all these properties are obeyed by the solution of the recursion relation `\eqref{eq:Recursion}`. Nonetheless, we have verified by explicit calculation that they do indeed hold. Details of this calculation will appear elsewhere.

```
\subsection{Concrete examples}
```

From `\eqref{eq:Recursion}`, the 3-point to 6-point single-minus stripped amplitudes are, using

```

\sg_{i,jk}=\sg\pa{\br{\tlam_i,\tlam_j+\tlam_k}},
\emph{etc.},

```

```

\begin{align}
A_{\{123\}}&=\sg_{\{12\}},\\
A_{\{1234\}}&=\frac{1}{
\{2\}\pa{\sg_{\{23\}}\sg_{\{41\}}+\sg_{\{12\}}\sg_{\{34\}}};\\
A_{\{12345\}}&=\frac{1}{
\{4\}\Big[&\sg_{\{51\}}\sg_{\{34\}}\sg_{\{2,34\}}+\sg_{\{51\}}\sg_{\{23\}}\s
g_{\{23,4\}}\\
&-
\sg_{\{51\}}\sg_{\{2,34\}}\sg_{\{23,4\}}+\sg_{\{45\}}\sg_{\{23\}}\sg_{\{1,23
\}}\notag\\
&+\sg_{\{45\}}\sg_{\{12\}}\sg_{\{12,3\}}-
\sg_{\{45\}}\sg_{\{1,23\}}\sg_{\{12,3\}}\notag\\
&+\sg_{\{51\}}\sg_{\{45\}}\sg_{\{12,34\}}+\sg_{\{12\}}\sg_{\{34\}}\sg_{\{12
,34\}}\Big]\notag,
\end{align}

```

```

\begin{widetext}
\begin{align}
\label{eq:A123456}
A_{\{123456\}}&=\frac{1}{\{8\}}\Big[
&-\sg_{\{1,23\}}\sg_{\{12,3\}}\sg_{\{123,4\}}\sg_{\{56\}}
+\sg_{\{1,23\}}\sg_{\{123,4\}}\sg_{\{23\}}\sg_{\{56\}}

```

$$\begin{aligned}
& +|\text{sg}_{\{1,234\}}\text{sg}_{\{12,34\}}\text{sg}_{\{123,4\}}\text{sg}_{\{56\}} \\
& -|\text{sg}_{\{1,234\}}\text{sg}_{\{12,34\}}\text{sg}_{\{34\}}\text{sg}_{\{56\}}\text{notag}\| \\
& \& -|\text{sg}_{\{1,234\}}\text{sg}_{\{123,4\}}\text{sg}_{\{23\}}\text{sg}_{\{56\}} \\
& -|\text{sg}_{\{1,234\}}\text{sg}_{\{2,34\}}\text{sg}_{\{23,4\}}\text{sg}_{\{56\}} \\
& +|\text{sg}_{\{1,234\}}\text{sg}_{\{2,34\}}\text{sg}_{\{34\}}\text{sg}_{\{56\}} \\
& +|\text{sg}_{\{1,234\}}\text{sg}_{\{23\}}\text{sg}_{\{23,4\}}\text{sg}_{\{56\}}\text{notag}\| \\
& \& +|\text{sg}_{\{12\}}\text{sg}_{\{12,3\}}\text{sg}_{\{123,4\}}\text{sg}_{\{56\}} \\
& -|\text{sg}_{\{12\}}\text{sg}_{\{12,34\}}\text{sg}_{\{123,4\}}\text{sg}_{\{56\}} \\
& +|\text{sg}_{\{12\}}\text{sg}_{\{12,34\}}\text{sg}_{\{34\}}\text{sg}_{\{56\}} \\
& +|\text{sg}_{\{12\}}\text{sg}_{\{345,6\}}\text{sg}_{\{45,6\}}\text{sg}_{\{56\}}\text{notag}\| \\
& \& -|\text{sg}_{\{1,23\}}\text{sg}_{\{12,3\}}\text{sg}_{\{45\}}\text{sg}_{\{45,6\}} \\
& +|\text{sg}_{\{1,23\}}\text{sg}_{\{23\}}\text{sg}_{\{45\}}\text{sg}_{\{45,6\}} \\
& +|\text{sg}_{\{12\}}\text{sg}_{\{12,3\}}\text{sg}_{\{45\}}\text{sg}_{\{45,6\}} \\
& -|\text{sg}_{\{12\}}\text{sg}_{\{3,45\}}\text{sg}_{\{34,5\}}\text{sg}_{\{345,6\}}\text{notag}\| \\
& \& +|\text{sg}_{\{12\}}\text{sg}_{\{3,45\}}\text{sg}_{\{345,6\}}\text{sg}_{\{45\}} \\
& +|\text{sg}_{\{12\}}\text{sg}_{\{34\}}\text{sg}_{\{34,5\}}\text{sg}_{\{345,6\}} \\
& -|\text{sg}_{\{2,34\}}\text{sg}_{\{23,4\}}\text{sg}_{\{234,5\}}\text{sg}_{\{61\}} \\
& +|\text{sg}_{\{2,34\}}\text{sg}_{\{234,5\}}\text{sg}_{\{34\}}\text{sg}_{\{61\}}\text{notag}\| \\
& \& +|\text{sg}_{\{2,345\}}\text{sg}_{\{23,45\}}\text{sg}_{\{234,5\}}\text{sg}_{\{61\}} \\
& -|\text{sg}_{\{2,345\}}\text{sg}_{\{23,45\}}\text{sg}_{\{45\}}\text{sg}_{\{61\}} \\
& -|\text{sg}_{\{2,345\}}\text{sg}_{\{234,5\}}\text{sg}_{\{34\}}\text{sg}_{\{61\}} \\
& -|\text{sg}_{\{2,345\}}\text{sg}_{\{3,45\}}\text{sg}_{\{34,5\}}\text{sg}_{\{61\}}\text{notag}\| \\
& \& +|\text{sg}_{\{2,345\}}\text{sg}_{\{3,45\}}\text{sg}_{\{45\}}\text{sg}_{\{61\}} \\
& +|\text{sg}_{\{2,345\}}\text{sg}_{\{34\}}\text{sg}_{\{34,5\}}\text{sg}_{\{61\}} \\
& +|\text{sg}_{\{23\}}\text{sg}_{\{23,4\}}\text{sg}_{\{234,5\}}\text{sg}_{\{61\}} \\
& -|\text{sg}_{\{23\}}\text{sg}_{\{23,45\}}\text{sg}_{\{234,5\}}\text{sg}_{\{61\}}\text{notag}\| \\
& \& +|\text{sg}_{\{23\}}\text{sg}_{\{23,45\}}\text{sg}_{\{45\}}\text{sg}_{\{61\}} \\
& +|\text{sg}_{\{345,6\}}\text{sg}_{\{45\}}\text{sg}_{\{45,6\}}\text{sg}_{\{61\}} \\
& +|\text{sg}_{\{23\}}\text{sg}_{\{45,6\}}\text{sg}_{\{56\}}\text{sg}_{\{61\}} \\
& +|\text{sg}_{\{34\}}\text{sg}_{\{345,6\}}\text{sg}_{\{56\}}\text{sg}_{\{61\}}\text{Big}.
\end{aligned}$$

\end{align}

\end{widetext}

\footnotetext{\label{SoftTheoremFootnote}Through

\eqref{eq:Ansatz}, this is seen to follow from the

conventional form  $\mathcal{A}_n \rightarrow \frac{\langle \text{av}\{1r\} \langle \text{av}\{nr\} \rangle \frac{[n,1]}{[p_{n,1}]^2 + i\epsilon} - \frac{\langle \text{av}\{n-1,r\} \langle \text{av}\{nr\} \rangle \frac{[n,n-1]}{[p_{n,n-1}]^2 + i\epsilon} \rangle \mathcal{A}_{n-1}$ .

Clearly a more concise formula is needed!

\section{Amplitudes in the first region}

\label{sec:Formula}

This section presents the next main result of this paper: a simple formula for the  $n$ -point single-minus amplitudes \eqref{eq:Recursion} with partially restricted

kinematics within the half-collinear regime.

\subsection{Restricted kinematics within the half-collinear regime}

We define the kinematic region  $\mathcal{SD}$  by the condition that, with  $\tilde{\alpha} = \omega(1, \tilde{z})$ , there exists at least one  $\mathsf{SO}(2,2)$  frame in which

$$\begin{aligned} & \text{\label{eq:Pn}} \\ & \omega_1 < 0, \text{\quad} \\ & \omega_a > 0, \text{\quad} \\ & a \in \{2, \dots, n\}. \end{aligned}$$

\end{align}

This region is fully consistent with the half-collinear regime where all  $v_{ij} = 0$  and further restricts the kinematics within that regime.

Unlike in Minkowski signature, we note that in Klein signature, there is no invariant meaning to a particle having positive frequency.

Nonetheless,  $\mathcal{SD}$  is  $\mathsf{SO}(2,2)$ -invariant because we only ask that there exists *some* frame in which  $\text{\label{eq:Pn}}$  holds.

Geometrically, it amounts to requiring that there exists some straight line through the origin in  $\mathbb{R}^2$  such that  $\tilde{\alpha}_1$  lies on one side and all other  $\tilde{\alpha}_a$  lie on the other.

$\mathcal{SD}$  describes a single ingoing self-dual gluon decaying to  $n-1$  outgoing anti-self-dual gluons, where ingoing/outgoing refers to the frame in which the inequalities  $\text{\label{eq:Pn}}$  hold.

Interestingly, we will see that the amplitudes dramatically simplify in  $\mathcal{SD}$ , where certain sign functions become independent of the frequencies  $\omega_k$ .

In particular,

$$\begin{aligned} & \text{\label{eq:SignSimplification}} \\ & \text{sg}_{ij} = \text{sg} \tilde{z}_{ij}, \text{\quad} \\ & \text{sg}_{1j} = \text{sg} \tilde{z}_{j1} \text{\quad} \\ & \text{for all } i, j \geq 2. \end{aligned}$$

\end{align}

Note however that the  $\omega_k$  cannot be eliminated from expressions such as  $\text{sg}_{2,34}$ .

\subsection{Concrete examples}

In region  $\mathcal{SD}$ , using [\eqref{eq:SignSimplification}](#), momentum conservation and spinor identities, one may show that the long expressions of the previous section dramatically simplify to

$$\begin{aligned}
 & \text{\label{eq:PTT3}} \\
 & A_{\{123\}}|_{\mathcal{SD}} = \frac{1}{2} \text{pa}\{\text{sg}_{\{12\}} + \text{sg}_{\{23\}}\}, \\
 & A_{\{1234\}}|_{\mathcal{SD}} = \frac{1}{4} \text{pa}\{\text{sg}_{\{12\}} + \text{sg}_{\{23\}}\} \text{pa}\{\text{sg}_{\{34\}} + \text{sg}_{\{41\}}\}, \\
 & A_{\{12345\}}|_{\mathcal{SD}} = \frac{1}{8} \text{pa}\{\text{sg}_{\{12\}} + \text{sg}_{\{23\}}\} \text{pa}\{\text{sg}_{\{34\}} + \text{sg}_{\{1,23\}}\} \\
 & \quad \&\text{\quad\quad\quad}\text{pa}\{\text{sg}_{\{45\}} + \text{sg}_{\{51\}}\}, \text{\notag} \\
 & \text{\label{eq:PTT6}} \\
 & A_{\{123456\}}|_{\mathcal{SD}} = \frac{1}{16} \text{pa}\{\text{sg}_{\{12\}} + \text{sg}_{\{23\}}\} \text{pa}\{\text{sg}_{\{34\}} + \text{sg}_{\{1,23\}}\} \\
 & \quad \&\text{\quad\quad\quad}\text{pa}\{\text{sg}_{\{45\}} + \text{sg}_{\{1,234\}}\} \text{pa}\{\text{sg}_{\{56\}} + \text{sg}_{\{61\}}\} \\
 & \quad \text{\notag} \\
 & \text{\end{align}}
 \end{aligned}$$

This suggests the possibility that there may exist a shorter formula for all  $n$ .

[\subsection{General formula}](#)

A conjecture that extends the pattern [\eqref{eq:PTT3}](#) to all  $n$ -particle amplitudes in region  $\mathcal{SD}$  is

$$\begin{aligned}
 & \text{\label{eq:Simplification}} \\
 & A_{\{1 \cdots n\}}|_{\mathcal{SD}} = \frac{1}{2^{n-2}} \prod_{m=2}^{n-1} \text{pa}\{\text{sg}_{\{m, m+1\}} + \text{sg}_{\{1, 2 \cdots m\}}\}. \\
 & \text{\end{align}}
 \end{aligned}$$

How sensible is this guess?

Regarding requirements [\eqref{eq:Cyclicity}](#), [\eqref{eq:softm}](#), the region  $\mathcal{SD}$  is clearly not cyclically invariant, as it singles out particle 1.

However, we can trivially construct a cyclically invariant answer by using cyclicity to extend [\eqref{eq:Simplification}](#) to other regions  $\mathcal{R}_k$  where only particle  $k$  has  $\omega_k < 0$ .

The remaining four conditions, when combined with cyclicity, do impose quite nontrivial constraints within  $\mathcal{SD}$ .

These constraints are all obeyed by our solution

`\eqref{eq:Simplification}`.

In fact, in this form, it is direct to check the soft theorem

`\eqref{eq:softm}` in the last label, and with some more

work, in any other label but 1.

The simplified form `\eqref{eq:Recursion}` has a simple

interpretation: in  $\mathcal{SD}$ , each factor is  $\frac{1}{2}$

$(\pm 1 \pm 1) \in \{-1, 0, 1\}$ , so  $A_{1 \cdots n|_{\mathcal{SD}}}$  is

piecewise-constant and jumps across codimension-one walls where the relevant brackets change sign.

The product form simply makes those walls explicit.

The rest of this work is devoted to proving that the

conjecture `\eqref{eq:Simplification}` is correct.

The proof uses ideas from time-ordered perturbation

theory and proceeds in three parts.

First, one shows that in  $\mathcal{SD}$ ,

`\begin{align}`

`\label{eq:V0inR1}`

$V_{\{t_{\text{lam}_2} \cdots t_{\text{lam}_n}\} \Big|_{\mathcal{SD}}} = 0,$

`\end{align}`

whilst  $\bar{V}_{\{t_{\text{lam}_2} \cdots t_{\text{lam}_n}\}}$  remains

nonzero.

Second, one must show that with `\eqref{eq:V0inR1}`, the

recursion `\eqref{eq:Recursion}` collapses to become

`\begin{align}`

`\label{eq:Collapse}`

$A_{1 \cdots$

$n \Big|_{\mathcal{SD}} = \bar{V}_{\{t_{\text{lam}_2} \cdots t_{\text{lam}_n}\} \Big|_{\mathcal{SD}}}.$

`\end{align}`

Finally, one must show that in  $\mathcal{SD}$ ,

$\bar{V}_{\{t_{\text{lam}_2} \cdots t_{\text{lam}_n}\}}$  reduces to our final

formula `\eqref{eq:Simplification}`. We briefly sketch each step below.

`\subsubsection{Vanishing of  $\bar{V}$ }`

We wish to show that within  $\mathcal{SD}$ ,

`\begin{align}`

`\label{eq:Vanish}`

$V_{\{t_{\text{lam}_2} \cdots t_{\text{lam}_n}\}} = 0$

`\end{align}`

for  $n \geq 3$ .

This can be interpreted as a causality

condition `\footnote{It is very reminiscent of the largest-`

time equation in time-ordered perturbation theory

$\text{\cite{Caron-Huot:2010fvq}.}$

in  $\text{\$SD\$}$  in the frame in which  $\text{\eqref{eq:Pn}}$  holds.

Importantly, all the  $\text{\$\omega\$\$}$ 's appearing in this expression are positive.

This will force at least one of the arguments of the

$\text{\Theta}$ -functions to be negative, implying

$\text{\eqref{eq:Vanish}}$ .

For each cut  $\text{\$j\$}$ , write left- and right-partial sums as

$\text{\begin{align}}$

$$\text{\tlam}_L = \sum_{a=2}^j \text{\tlam}_a = \Omega_L(1, \tilde{z}_L),$$

$$\text{\tlam}_R = \sum_{a=j+1}^n \text{\tlam}_a = \Omega_R(1, \tilde{z}_R),$$

$\text{\end{align}}$

with  $\text{\$\Omega_{L,R} > 0\$}$  and  $\text{\$\tilde{z}_{L,R}\$}$  weighted averages of the  $\text{\$\tilde{z}\$}$ 's on each side.

Then

$\text{\begin{align}}$

$$\text{\br{\tlam}_R \text{\tlam}_L} = \Omega_L \Omega_R \text{\tilde{z}_{R,L}},$$

$$\text{\br{\tlam}_{j+1} \text{\tlam}_j} = \omega_j \omega_{j+1} \text{\tilde{z}_{j+1,j}}.$$

$\text{\end{align}}$

Since  $\text{\$\Omega_L \Omega_R > 0\$}$  and

$\text{\$\omega_j \omega_{j+1} > 0\$}$ , the sign of the ratio in the

$\text{\Theta}$ -factor

in  $\text{\eqref{eq:Vertex}}$  is the sign of

$$\text{\tilde{z}_{R,L} / \tilde{z}_{j+1,j}}.$$

A weighted-variance identity implies that as  $\text{\$j\$}$  runs

from  $\text{\$2\$}$  to  $\text{\$n\$}$ , there must be at least one cut  $\text{\$j^{\star}\$}$

for which  $\text{\$\tilde{z}_{R,L}\$}$  has the **same sign** as

the adjacent increment  $\text{\$\tilde{z}_{j+1,j}\$}$ .

For that  $\text{\$j^{\star}\$}$ , the ratio is positive, so the

corresponding factor  $\text{\Theta(-\text{\text{positive}}) = 0\$}$ , and

thus the whole product in  $\text{\eqref{eq:Vertex}}$  vanishes.

$\text{\subsubsection{Collapsing the recursion}}$

We have in fact shown something slightly more general:

for **every** consecutive  $\text{\$S \subset \{2, \dots, n\}\$}$

with  $\text{\$|S| \ge 2\$}$ ,

$$\begin{aligned} & V_{\{\tilde{\lambda}_{S_1} \cdots \tilde{\lambda}_{S_k}\}} = 0. \\ & \end{aligned}$$

Then, from [\eqref{eq:Preamplitudes}](#), we find that on  $\mathcal{SD}$ ,

$$\begin{aligned} & \label{eq:Abar0} \\ & \bar{A}_S \Big|_{\mathcal{SD}} = 0, \\ & \end{aligned}$$

while singletons remain,  $\bar{A}_i = 1$ .

This collapses the recursion.

Using the cyclicity of the color-ordered amplitude, we may write

$$\begin{aligned} & \label{eq:Abar0} \\ & A_{\{1 \cdots n\}} = A_{\{2 \cdots n1\}}. \\ & \end{aligned}$$

Now apply the recursion [\eqref{eq:Recursion}](#) to  $A_{\{2 \cdots n1\}}$ , *\emph{i.e.}*, partition the ordered set  $(2, 3, \dots, n)$ .

By [\eqref{eq:Abar0}](#), the only nonzero contribution is the all-singleton partition  $(2|3|\cdots|n)$ , so

$$\begin{aligned} & \label{eq:AtoPTc} \\ & A_{\{2 \cdots n1\}} \Big|_{\mathcal{SD}} = -\widehat{\text{rm PT}}_{\{\text{tlam}_2 \text{tlam}_3 \cdots \text{tlam}_n\}} \Big|_{\mathcal{SD}}. \\ & \end{aligned}$$

But since  $V_{\{\text{tlam}_2 \cdots \text{tlam}_n\}} = 0$ , we have  $\widehat{\text{rm PT}} = V - \bar{V} = -\bar{V}$  for this list.

Therefore,

$$\begin{aligned} & \label{eq:AequalsVbar} \\ & A_{\{1 \cdots n\}} \Big|_{\mathcal{SD}} = A_{\{2 \cdots n1\}} \Big|_{\mathcal{SD}} = \bar{V}_{\{\text{tlam}_2 \cdots \text{tlam}_n\}}. \\ & \end{aligned}$$

This proves the collapse of the recursion [\eqref{eq:Recursion}](#) to [\eqref{eq:Collapse}](#).

[\subsubsection{Evaluating](#)

[\texorpdfstring{\bar{V}\\_{\{\text{tlam}\\_2 \cdots \text{tlam}\\_n\}}}{V}}](#)

Last but not least, we reorganize the vertex in terms of sign functions.

Recall that  $\bar{V}_{\{\text{tlam}_2 \cdots \text{tlam}_n\}}$  is defined as

$$\begin{aligned} & \label{eq:VbarStart} \end{aligned}$$

$$\bar{V}_{\{t_2 \cdots t_n\}} = \prod_{m=2}^{n-1} \text{sg}_{\{m, m+1\}} \Theta \left[ \frac{\bar{t}_2 \cdots t_m}{\bar{t}_m \cdots t_{m+1}} \right].$$

By momentum conservation,  $\bar{t}_{m+1 \cdots n} = -\bar{t}_1 - \bar{t}_2 \cdots t_m$ , and using the antisymmetry of the bracket,  $\bar{V}$  becomes

$$\bar{V}_{\{t_2 \cdots t_n\}} = \prod_{m=2}^{n-1} \text{sg}_{\{m, m+1\}} \Theta \left[ \frac{\bar{t}_1 \bar{t}_2 \cdots t_m}{\bar{t}_m \cdots t_{m+1}} \right].$$

$\bar{V}$  after

end align

Using the relation between the  $\text{sg}$  and  $\Theta$  functions, one readily finds that

$$\bar{V}_{\{t_2 \cdots t_n\}} = \frac{1}{2^{n-2}} \prod_{m=2}^{n-1} \text{sg}_{\{m, m+1\}} \left[ \frac{\bar{t}_1 \bar{t}_2 \cdots t_m}{\bar{t}_m \cdots t_{m+1}} \right].$$

end align

Combining  $\bar{V}$  with

$\bar{V}$  recovers exactly our final result  $\bar{V}$ .

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\appendix  
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\section{The master identity}  
 \label{app:Master}

Consider the well-known identity

\begin{align}  
 \label{eq:Identity2}  
 \br{\frac{a\_1}{(b\_2+i\epsilon)}+\frac{a\_2}{(b\_1+i\epsilon)}}\delta(a\_1b\_1+a\_2b\_2)=-\frac{i}{2}\br{\sg(a\_1)+\sg(a\_2)}\delta(b\_1)\delta(b\_2),  
 \end{align}

which readily follows from writing  $\frac{1}{b+i\epsilon}=\mathrm{PV}\frac{1}{b}-\frac{i}{2}\delta(b)$ , where the  $\delta$ -function is normalized as in \eqref{eq:DeltaNormalization}.

In this appendix, we use manipulations from time-ordered perturbation theory to derive a powerful generalization of this identity:

\begin{align}  
 \label{eq:Master}  
 &\phantom{=}\delta\pa{\sum\_{k=1}^na\_kb\_k}\sum\_{i=1}^n\frac{a\_i}{\prod\_{j\neq i}(b\_j+i\epsilon)}\notag\\
 &=\frac{1}{(2i)^{n-1}}\br{\sum\_{i\_1}\sg(a\_{i\_1})+\sum\_{i\_1<i\_2<i\_3}\sg(a\_{i\_1}a\_{i\_2}a\_{i\_3})+\sum\_{i\_1<\cdots<i\_5}\sg(a\_{i\_1}a\_{i\_2}a\_{i\_3}a\_{i\_4}a\_{i\_5})+\ldots}\prod\_{i=1}^n\delta(b\_i)  
 .  
 \end{align}

For example, when  $n=3$ , this generalizes the identity

\eqref{eq:Identity2} to  
 \begin{align}

\text{LHS}&=\delta\pa{\sum\_{i=1}^3a\_ib\_i}\br{\frac{a\_1}{(b\_2+i\epsilon)(b\_3+i\epsilon)}+\frac{a\_2}{(b\_1+i\epsilon)(b\_3+i\epsilon)}+\frac{a\_3}{(b\_1+i\epsilon)(b\_2+i\epsilon)}}\\
 &=\text{RHS}&=-\frac{1}{4}\br{\sg(a\_1)+\sg(a\_2)+\sg(a\_3)+\sg(a\_1)\sg(a\_2)\sg(a\_3)}\delta(b\_1)\delta(b\_2)\delta(b\_3).  
 \end{align}

To establish the generalized identity \eqref{eq:Master}, we prove its Fourier transform, that is, the corresponding

identity in the time domain where we may think of the  $b_i$  as energies.

We then have the following:

$$\begin{aligned} & \int \prod_{k=1}^n b_k e^{i \sum_{k=1}^n t_k} \delta \left( \sum_{k=1}^n a_k - \prod_{j \neq i} (b_j + i\epsilon) \right) \\ & = \sum_{i=1}^n \int \prod_{k=1}^n b_k e^{i \sum_{k=1}^n t_k} \delta \left( \sum_{k=1}^n a_k - \prod_{j \neq i} (b_j + i\epsilon) \right) \\ & = (-2\pi i)^{n-1} \sum_{i=1}^n \int_{-\infty}^{\infty} \prod_{k=1}^n b_k e^{i \sum_{k=1}^n t_k} \delta \left( \sum_{k=1}^n a_k - \prod_{j \neq i} (b_j + i\epsilon) \right) \\ & \quad \times \Theta(-t_i + \sum_{j \neq i} a_j) \\ & = (-2\pi i)^{n-1} (2\pi i) \int_{-\infty}^{\infty} \prod_{k=1}^n b_k e^{i \sum_{k=1}^n t_k} \delta \left( \sum_{k=1}^n a_k - \prod_{j \neq i} (b_j + i\epsilon) \right) \\ & \quad \times \Theta(-t_i + \sum_{j \neq i} a_j) \\ & = (-2\pi i)^{n-1} (2\pi i) \prod_{i=1}^n \Theta(-t_i + \sum_{j \neq i} a_j) \\ & \quad \times \left[ \prod_{i=1}^n \Theta(a_i) - \prod_{i=1}^n \Theta(-a_i) \right]. \end{aligned}$$

Finally, Fourier transforming in  $t_i$  yields

$$\begin{aligned} & \delta \left( \sum_{k=1}^n a_k - \prod_{j \neq i} (b_j + i\epsilon) \right) \\ & = i^{1-n} \prod_{i=1}^n \Theta(a_i) \delta(b_i) - i^{1-n} \prod_{i=1}^n \Theta(-a_i) \delta(b_i), \end{aligned}$$

which recovers the master identity after using

$$\Theta(x) = \frac{1 + \text{sg}(x)}{2}.$$

## Derivation of recursion relation

`\label{app:Derivation}`

This appendix derives the Berends--Giele recursion relation `\cite{Berends:1987me}` for the off-shell currents in Yang--Mills theory.

In turn, this implies a recursion formula for the planar form factors of the theory.

## Berends--Giele Recursion

In QFT,  $n$ -point scattering amplitudes can be computed from form factors  $\mathcal{F}_S$  with one leg off-shell and  $n-1$  legs on-shell.

For a color-ordered gluon amplitude, the ordering is inherited from the corresponding form factor.

To avoid notational clutter, we have here taken the last particle to be negative helicity (as opposed to the first one in the main text).

The final on-shell formula is insensitive to this choice.

We thus write

$$\begin{aligned} & \text{\label{eq:AmplitudeFormFactor}} \\ & \mathcal{A}_{\{1\cdots n\}} = \lim_{p_n^2 \rightarrow 0} - \\ & ip_n^2 \mathcal{F}_{\{1\cdots n-1\}}(\sum_{i=1}^n p_i, \quad \\ & p_n = -\sum_{i=1}^{n-1} p_i. \end{aligned}$$

These planar coefficients satisfy the Berends--Giele (BG) recursion, equivalent to summing over Feynman diagrams with one leg off-shell.

When the rest of the (on-shell) legs are plus-helicity gluons, it is known that this recursion is equal to the one in Self-Dual Yang--Mills theory (SDYM)

\cite{Cangemi:1996rx,10.1143/PTPS.123.1}, which reads

$$\begin{aligned} & \text{\label{eq:FormFactor}} \\ & \mathcal{F}_{\{1\cdots m\}} = \frac{1}{p_{\{1\cdots m\}}^2 + i\epsilon} \sum_{j=1}^{m-1} \text{br}\{\text{tlam}_{\{1\cdots j\}} \text{tlam}_{\{j+1\cdots m\}}\} \mathcal{F}_{\{1\cdots j\}} \\ & \text{\,}, \mathcal{F}_{\{j+1\cdots m\}}, \end{aligned}$$

where  $p_{\{1\cdots m\}} = \sum_{i=1}^m p_i$  and

$\text{tlam}_{\{1\cdots m\}} = \sum_{i=1}^m \text{tlam}_i$ .

This is essentially the equation of motion of the theory and determines its classical solutions.

### \subsection{General form factor}

The form factor recursion \eqref{eq:FormFactor} is solved by means of our two-dimensional preamplitudes  $\bar{A}_S$  in \eqref{eq:Preamplitudes}, by replacing one vertex by  $PT$  where  $PT$  refers to the incomplete Parke--Taylor factor \eqref{eq:PTC}. Let  $(S_1|\cdots|S_A)$  be an ordered partition of the word  $(1\cdots m)$ , and write the block momenta

$$K_a = \sum_{i \in S_a} t_{lam_i}.$$

We claim the solution to the recursion  $\sim$  [\eqref{eq:FormFactor}](#) is

$$\begin{aligned} & \text{\label{eq:FormFactorSolution}} \\ & \mathcal{F}_{1 \cdots m} = \sum_{\text{mathrm{o.p.}} \text{mathrm{PT}}_{K_1 \cdots K_A}} \prod_{a=1}^A \text{Big}(\bar{A}_{S_a}, \delta_{S_a} \text{Big}), \end{aligned}$$

where the sum is over all possible ordered partitions of  $(1 \cdots m)$  into  $A$  blocks  $S_a$ , as well as over the number  $A=1, \dots, m$  of blocks. As in the main text,  $\delta_{S_a}$  denotes the product of  $\delta(z_{i,i+1})$  internal to the block  $S_a$ .

To see this, we first establish [\eqref{eq:PT}](#), which we quote again here for the benefit of the reader:

$$\begin{aligned} & \text{\label{eq:PT2}} \\ & \text{mathrm{PT}}_{1 \cdots n} - \delta_{1 \cdots n} \prod_{j=1}^n \frac{\text{br}\{t_{lam_{1 \cdots j}} t_{lam_{j+1 \cdots n}}\}}{\{p_{1 \cdots n}^{2+i\epsilon}\} \text{mathrm{PT}}_{1 \cdots j} \text{mathrm{PT}}_{j+1 \cdots n}}. \end{aligned}$$

Away from the half-collinear regime, we have  $\delta_{1 \cdots n} = 0$ , the  $\epsilon$ 's on the RHS may be neglected, and this identity is a standard algebraic identity for Parke-Taylor factors.

The  $\epsilon$ 's are important in the half-collinear limit and the general identity  $\sim$  [\eqref{eq:PT2}](#) follows from [\eqref{eq:Master}](#) with the values

$$\begin{aligned} & a_r = -\text{br}\{t_{lam_{1 \cdots r}} t_{lam_{r+1 \cdots n}}\} \prod_{\ell \neq r} \text{br}\{t_{lam_{\ell}} t_{lam_{\ell+1}}\}, \\ & b_r = p_{r,r+1}^2, \end{aligned}$$

for  $r=1, \dots, n-1$ , while  $a_n = \prod_{\ell} [\ell, \ell+1]$  and  $b_n = p_{1 \cdots n}^2$ .

Next, we will verify [\eqref{eq:FormFactorSolution}](#) is a solution by inserting it into the RHS of [\eqref{eq:FormFactor}](#). For each term in the sum over  $j$  in [\eqref{eq:FormFactor}](#), we now have sums over all ordered partitions  $(1 \cdots j) = (L_1 \cdots L_B)$  and

$$F_{j+1 \cdots m} = (R_1 \cdots R_C):$$

$\begin{aligned}$

$$\begin{aligned} & \mathcal{F}_{1 \cdots m} = \sum_{j=1}^{m-1} \sum_{\text{o.p.}} \frac{\text{tr} \{ \text{t} \}_{1 \cdots j}, \text{t} \}_{j+1 \cdots m} \}}{p^2_{1 \cdots m} + i\epsilon} \\ & \text{tr} \{ \text{PT} \}_{L_1 L_2 \cdots L_B} \text{tr} \{ \text{PT} \}_{R_1 R_2 \cdots R_C} \} \text{notag} \end{aligned}$$

$$\begin{aligned} & \text{tr} \{ \text{A} \}_{L_1} \cdots \text{tr} \{ \text{A} \}_{L_B} \} \text{tr} \{ \text{A} \}_{R_1} \cdots \text{tr} \{ \text{A} \}_{R_C} \} \delta_{L_1} \cdots \delta_{L_B} \delta_{R_1} \cdots \delta_{R_C}, \\ & \end{aligned}$$

where we have abused notation slightly, denoting both the set and its block momentum by the same letter  $L_a$  or  $R_b$ .

Note that  $\tilde{\lambda}_{1 \cdots j} = \sum_k \tilde{\lambda}_{L_k}$  and  $\tilde{\lambda}_{j+1 \cdots m} = \sum_k \tilde{\lambda}_{R_k}$ .

The second line depends only on the total partition  $(S_1 \cdots S_A) = (L_1 \cdots L_B | R_1 \cdots R_C)$ , with  $A = B + C$ , and does not know about the separation into 'left' and 'right'.

For a fixed such partition, the  $\mathcal{F}$ -sum is exactly the PT recursion  $\text{eq:PT2}$ : it produces (i) a PT for the combined partition, and (ii) a contact term  $\delta_{V}$ .

In equations, we have

$\begin{aligned}$

$$\begin{aligned} & \mathcal{F}_{1 \cdots m} = \sum_{\text{n.t.p.}} \text{tr} \{ \text{A} \}_{S_1} \cdots \text{tr} \{ \text{A} \}_{S_A} \} \delta_{S_1} \cdots \delta_{S_A} \\ & \text{tr} \{ \sum_{B+C=A} \frac{\text{tr} \{ \text{t} \}_{1 \cdots j}, \text{t} \}_{j+1 \cdots m} \}}{p^2_{1 \cdots m} + i\epsilon} \text{tr} \{ \text{PT} \}_{L_1 \cdots L_B} \text{tr} \{ \text{PT} \}_{R_1 \cdots R_C} \} \end{aligned}$$

$$\begin{aligned} & = \sum_{\text{n.t.p.}} \text{tr} \{ \text{A} \}_{S_1} \cdots \text{tr} \{ \text{A} \}_{S_A} \} \delta_{S_1} \cdots \delta_{S_A} \text{tr} \{ \text{PT} \}_{S_1 S_2 \cdots S_A} - \delta_{1 \cdots m} \\ & \sum_{\text{n.t.p.}} \text{tr} \{ \text{A} \}_{S_1} \cdots \text{tr} \{ \text{A} \}_{S_A} \} V_{S_1 S_2 \cdots S_A} \end{aligned}$$

$$\begin{aligned} & = \sum_{\text{n.t.p.}} \text{tr} \{ \text{A} \}_{S_1} \cdots \text{tr} \{ \text{A} \}_{S_A} \} \delta_{S_1} \cdots \delta_{S_A} \text{tr} \{ \text{PT} \}_{S_1 S_2 \cdots S_A} + \delta_{1 \cdots m} \text{tr} \{ \text{A} \}_{1 \cdots m}, \end{aligned}$$

\end{align}

where n.t.p. denotes nontrivial partitions (having more than one block).

The PT piece reproduces the nontrivial  $A$ -block contribution on the LHS of

\eqref{eq:FormFactorSolution}.

The contact piece  $\delta_V$  precisely stitches blocks together by a  $V$ -vertex, and the resulting sum over nontrivial partitions is the recursion~\eqref{eq:Preamplitudes} for the preamplitudes.

The final term in the third line is the missing 1-term partition, thus recovering \eqref{eq:FormFactorSolution}.

\subsection{LSZ Reduction}

The single-minus amplitude is obtained by putting the 'last' leg on-shell, as in

\eqref{eq:AmplitudeFormFactor}.

On the support of the collinear  $\delta$ -functions inside each block of the form factor

\eqref{eq:FormFactorSolution}, every block momentum  $K_a$  is null.

The only remaining singular factor is the PT term associated to adjacent channels.

The on-shell limit of a PT factor is evaluated using the master identity in App.~\ref{app:Master}, resulting in

\begin{align}

$$\begin{aligned} & \text{\label{eq:LSZ}} \\ & \lim_{p_n^2 \rightarrow 0} p_n^2 \text{\mathrm{PT}}_{K_1 \cdots K_k} \delta^4(p_n) \text{\mathrm{PT}}_{K_1 \cdots K_k} \\ & = \widehat{\text{\mathrm{PT}}}_{K_1 \cdots K_k} \delta^4(p_n) \\ & = (V_{K_1 \cdots K_k} - \bar{V}_{K_1 \cdots K_k}) \delta^4(p_n) \\ & = \text{\mathrm{PT}}_{K_1 \cdots K_k} \delta^4(p_n) \text{\mathrm{PT}}_{K_1 \cdots K_k} \end{aligned}$$

\end{align}

After stripping off the universal momentum-conservation support (here,  $\sum_{i=1}^n t_{\lambda_i} = 0$ , so  $t_{\lambda_n} = -\sum_{i=1}^{n-1} t_{\lambda_i}$ ), \eqref{eq:FormFactorSolution} and \eqref{eq:LSZ} yield, after some algebra, the final result \eqref{eq:Recursion}:

\begin{align}

$$A_{1 \cdots n} = -\sum_{\substack{k=1 \\ k \leq n-1}} A_{1 \cdots k} S_{k+1 \cdots n}$$

```

1}}\widehat{\mathrm{PT}}_{\{t_{lam\_S_1}\}\cdots\{t_{lam\_S_k}\}}
\;\prod_{a=1}^k\bar{A}_{S_a}\;,
\end{align}

\bibliographystyle{utphys}
\bibliography{SMGA.bib}

\end{document}

```

👉 I'll focus on Appendix A's master identity and Appendix B's BG/LSZ derivation, re-deriving the key distributional steps (including all  $i\epsilon$ ,  $\delta$ ,  $\Theta$ , and sign conventions) and then tracing precisely how they feed into Eqs. (2.25) and (3.8).

Thought for 30m 32s >

## What Appendices A and B are doing, in one line

Appendix A gives a **distributional replacement** for a naive partial-fraction/algebraic cancellation when several denominators simultaneously pinch; Appendix B uses that replacement twice: once to get the corrected Parke–Taylor “recursion” with a **contact term** and once in LSZ to turn the off-shell form factor into the on-shell single-minus recursion.

## Appendix A: the master identity (precise distribution manipulations)

### Conventions that matter

They normalize delta-functions by

$$\int \delta(x) dx = 2\pi,$$

(1.1)

so (equivalently) the discontinuity of the Feynman denominator is

$$\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon} = -i\delta(x),$$

(1.2)

and therefore

$$\frac{1}{x+i\epsilon} = \text{PV} \frac{1}{x} - \frac{i}{2} \delta(x).$$

(1.3)

A subtle but crucial corollary: **the Heaviside step function is still the standard one** (jump 1), so

$$\frac{d}{dx} \Theta(x) = \delta_{\text{std}}(x) = \frac{1}{2\pi} \delta(x).$$

(1.4)

That "1/(2π)" is exactly where the extra 2π later appears in their Fourier-domain derivation.

---

### The master identity in the cleanest form

They state it as a sum over odd products of sign functions, but the most compact equivalent statement is

$$\delta\left(\sum_{k=1}^n a_k b_k\right) \sum_{i=1}^n \frac{a_i}{\prod_{j \neq i} (b_j + i\epsilon)} = i^{1-n} \left( \prod_{k=1}^n \Theta(a_k) - \prod_{k=1}^n \Theta(-a_k) \right) \prod_{k=1}^n \delta(b_k).$$

(1.5)

Expanding  $\Theta(x) = \frac{1+\text{sg}(x)}{2}$  turns the RHS into exactly their "odd subsets" sign-sum with overall factor  $1/(2i)^{n-1}$ .

---

### Proof logic (Fourier transform → step functions → boundary term)

Start from the Fourier transform in the variables  $b_k$ :

$$I(t) \equiv \int d^n b e^{i \sum_k t_k b_k} \delta\left(\sum_k a_k b_k\right) \sum_{i=1}^n \frac{a_i}{\prod_{j \neq i} (b_j + i\epsilon)}.$$

(1.6)

**Step 1: represent the delta constraint.** With their delta normalization,

$$\delta(x) = \int_{-\infty}^{\infty} d\gamma e^{-i\gamma x}.$$

(1.7)

So the delta inserts a common shift  $t_k \mapsto t_k - \gamma a_k$ .

**Step 2: do the  $b$ -integrals.**

- For the one index  $i$  without a propagator factor, you get

$$\int db_i e^{i(t_i - \gamma a_i)b_i} = \delta(t_i - \gamma a_i).$$

(1.8)

- For each  $j \neq i$ , you need

$$\int_{-\infty}^{\infty} db \frac{e^{iub}}{b + i\epsilon} = -2\pi i \Theta(-u), \quad u \in \mathbb{R}.$$

(1.9)

Putting these together gives

$$I(t) = (-2\pi i)^{n-1} \sum_{i=1}^n \int d\gamma a_i \delta(t_i - \gamma a_i) \prod_{j \neq i} \Theta(-t_j + \gamma a_j).$$

(1.10)

**Step 3: convert the delta to a  $\gamma$ -derivative of a step function.**

Using (1.4),

$$\frac{\partial}{\partial \gamma} \Theta(-t_i + \gamma a_i) = a_i \delta_{\text{std}}(t_i - \gamma a_i) = \frac{a_i}{2\pi} \delta(t_i - \gamma a_i),$$

so

$$a_i \delta(t_i - \gamma a_i) = 2\pi \frac{\partial}{\partial \gamma} \Theta(-t_i + \gamma a_i).$$

(1.11)

This is precisely why their next line acquires an explicit factor  $2\pi$ .

Then

$$I(t) = (-2\pi i)^{n-1} (2\pi) \int d\gamma \sum_{i=1}^n \frac{\partial}{\partial \gamma} \Theta(-t_i + \gamma a_i) \prod_{j \neq i} \Theta(-t_j + \gamma a_j).$$

(1.12)

But the sum is exactly the derivative of the full product:

$$\sum_i \left( \frac{\partial}{\partial \gamma} \Theta_i \right) \prod_{j \neq i} \Theta_j = \frac{\partial}{\partial \gamma} \prod_{k=1}^n \Theta(-t_k + \gamma a_k).$$

(1.13)

So you get a pure boundary term:

$$I(t) = (-2\pi i)^{n-1} (2\pi) \left[ \prod_{k=1}^n \Theta(-t_k + \gamma a_k) \right]_{\gamma=-\infty}^{\gamma=+\infty}.$$

(1.14)

As  $\gamma \rightarrow +\infty$ ,  $-t_k + \gamma a_k$  has the sign of  $a_k$ ; as  $\gamma \rightarrow -\infty$  it has the sign of  $-a_k$ .

Hence

$$I(t) = (-2\pi i)^{n-1} (2\pi) \left[ \prod_{k=1}^n \Theta(a_k) - \prod_{k=1}^n \Theta(-a_k) \right].$$

(1.15)

**Step 4: invert the Fourier transform.** With the same conventions, inversion brings down a factor  $(2\pi)^{-n}$ , and  $\int dt e^{-itb} = \delta(b)$ . That yields exactly (1.5).

That's the entire content: the complicated LHS collapses to a product of delta functions  $\prod \delta(b_i)$  times a **pure sign/step-function coefficient** which is (in time-ordered language) the difference between "all  $a_i$  future-directed" and "all  $a_i$  past-directed".

## Appendix B: Berends–Giele $\rightarrow$ partition solution $\rightarrow$ LSZ $\rightarrow$ on-shell recursion

### B.1 BG recursion used

They take (SDYM/all-plus) BG recursion for a form factor with one off-shell leg:

$$\mathcal{F}_{1\dots m} = \frac{1}{p_{1\dots m}^2 + i\epsilon} \sum_{j=1}^{m-1} [\tilde{\lambda}_{1\dots j} \tilde{\lambda}_{j+1\dots m}] \mathcal{F}_{1\dots j} \mathcal{F}_{j+1\dots m}.$$

(1.16)

### B.2 The claimed closed-form solution and what it really encodes

They propose

$$\mathcal{F}_{1\dots m} = \sum_{\text{o.p.}} \text{PT}_{K_1\dots K_A} \prod_{a=1}^A (\bar{A}_{S_a} \delta_{S_a}), \quad K_a = \sum_{i \in S_a} \tilde{\lambda}_i,$$

(1.17)

i.e. **sum over ordered partitions** of the ordered word  $(1 \dots m) | \dots | S_A$ .

Interpretation:

- $\bar{A}_{S_a}$  is a purely 2d "preamplitude" encoding how an SDYM multi-point contact vertex glues subcurrents.
- $\delta_{S_a}$  enforces half-collinearity *inside* the block.
- $\text{PT}_{K_1 \dots K_A}$  is the "skeleton" connecting blocks by adjacent propagators.

The reason this solves BG recursion is a corrected "PT recursion identity" (their (B.6)/(PT2)).

### B.3 The key intermediate identity: corrected PT recursion (where Appendix A enters)

They need (I rewrite their (PT2))

$$\text{PT}_{1 \dots n} - \delta_{1 \dots n} V_{\tilde{\lambda}_1 \dots \tilde{\lambda}_n} = \sum_{j=1}^{n-1} \frac{[\tilde{\lambda}_{1 \dots j} \tilde{\lambda}_{j+1 \dots n}]}{p_{1 \dots n}^2 + i\epsilon} \text{PT}_{1 \dots j} \text{PT}_{j+1 \dots n}.$$

(1.18)

Away from the half-collinear locus,  $\delta_{1 \dots n} = 0$  and you recover the standard algebraic partial-fraction identity. On the half-collinear locus, the **contact term**  $\delta V$  is exactly what the master identity produces.

#### How the master identity is matched (precisely)

They apply Appendix A with

- $b_r = p_{r,r+1}^2$  for  $r = 1, \dots, n-1$ ,
- $b_n = p_{1 \dots n}^2$

and

$$a_r = -[\tilde{\lambda}_{1 \dots r} \tilde{\lambda}_{r+1 \dots n}] \prod_{\ell \neq r} [\tilde{\lambda}_\ell \tilde{\lambda}_{\ell+1}], \quad a_n = \prod_{\ell=1}^{n-1} [\tilde{\lambda}_\ell \tilde{\lambda}_{\ell+1}].$$

(1.19)

Two key facts then happen:

1. **The algebraic relation**  $\sum a_k b_k = 0$  is exactly the "chain decomposition" of the total invariant  $p_{1 \dots n}^2$  in the gauge  $\lambda_i = (1, z_i)$ , namely

$$p_{1 \dots n}^2 = \sum_{r=1}^{n-1} \langle r \ r+1 \rangle [\tilde{\lambda}_{1 \dots r} \tilde{\lambda}_{r+1 \dots n}], \quad \langle r \ r+1 \rangle = z_{r,r+1}.$$

(1.20)

Multiplying (1.20) by  $\prod_{\ell} [\ell, \ell+1]$  and using  $p_{r,r+1}^2 = \langle r r+1 \rangle [r r+1]$  gives  $\sum a_k b_k = 0$  with (1.19).

2. The master identity then tells you that the naive cancellation becomes a **distribution supported at  $b_r = 0$**  (i.e.  $\langle r r+1 \rangle = 0$ , the half-collinear wall), producing the  $\delta_{1 \dots n} V$  term. The sign/step-function coefficient turns into exactly

$$V_{\tilde{\lambda}_1 \dots \tilde{\lambda}_n} = \prod_{r=1}^{n-1} \text{sg}_{r,r+1} \Theta \left( - \frac{[\tilde{\lambda}_1 \dots \tilde{\lambda}_{r+1} \dots \tilde{\lambda}_n]}{[\tilde{\lambda}_r \tilde{\lambda}_{r+1}]} \right),$$

(1.21)

because  $a_r/a_n = -[\tilde{\lambda}_1 \dots \tilde{\lambda}_{r+1} \dots \tilde{\lambda}_n]/[\tilde{\lambda}_r \tilde{\lambda}_{r+1}]$  and  $\text{sg}(a_n) = \prod \text{sg}([\tilde{\lambda}_r \tilde{\lambda}_{r+1}])$ .

That is the exact "Appendix A  $\rightarrow$  corrected PT recursion" bridge.

#### B.4 Why the partition sum solves BG recursion (mechanism)

When you insert (1.17) into BG recursion (1.16), each split  $j$  induces:

- an ordered partition of the left word  $(1 \dots j)$ ,
- an ordered partition of the right word  $(j+1 \dots m)$ .

For any fixed **total** ordered partition

$$(1 \dots m) = (S_1 | \dots | S_A),$$

the sum over all split points  $j$  that cut between blocks reproduces exactly the RHS of (1.18), giving:

- the PT factor for the full partition  $PT_{S_1 \dots S_A}$ ,
- plus the contact term  $-\delta_{1 \dots m} V_{S_1 \dots S_A}$ .

That contact term is *exactly* what the preamplitude recursion for  $\bar{A}$  is designed to absorb:

$$\bar{A}_{1 \dots m} = - \sum_{\text{o.p., } A \geq 3} V_{S_1 \dots S_A} \prod_{a=1}^A \bar{A}_{S_a},$$

so  $-\sum V \prod \bar{A} = \bar{A}_{1 \dots m}$ , which reinstates the missing  $A = 1$  ("single block") term in (1.17).

This is why the solution works: **PT recursion glues the skeleton, and the contact term generates the vertex recursion for  $\bar{A}$ .**

## B.5 LSZ: where $\widehat{\text{PT}} = V - \bar{V}$ comes from

The amplitude is obtained by

$$\mathcal{A}_{1\dots n} = \lim_{p_n^2 \rightarrow 0} (-i) p_n^2 \mathcal{F}_{1\dots n-1} \delta^4 \left( \sum_{i=1}^n p_i \right).$$

(1.22)

On the support of the internal collinear  $\delta_{S_a}$ , each block momentum  $K_a$  is null, so the only nontrivial singular structure that survives LSZ sits in the adjacent-channel PT skeleton. Using the master identity again to take that on-shell limit yields their key reduction

$$\lim_{p_n^2 \rightarrow 0} p_n^2 \text{PT}_{K_1 \dots K_k} \delta^4 \left( \sum_{a=1}^k K_a + p_n \right) = \widehat{\text{PT}}_{K_1 \dots K_k} \delta_{1\dots k, n} \delta^2 \left( \sum_j \tilde{\lambda}_j \right),$$

(1.23)

with

$$\widehat{\text{PT}}_{K_1 \dots K_k} = V_{K_1 \dots K_k} - \bar{V}_{K_1 \dots K_k},$$

(1.24)

and  $\bar{V}$  is the same product as  $V$  but with  $\Theta(+\dots)$  instead of  $\Theta(-\dots)$ .

(Operationally: it's the same master-identity coefficient but with the opposite sign choice in the step-function argument, corresponding to the opposite  $i\epsilon$  routing that appears in the on-shell extraction.)

Finally, stripping off the universal support  $\delta_{1\dots n} \delta^2(\sum \tilde{\lambda})$  gives the on-shell recursion quoted in the main text:

$$A_{1\dots n} = - \sum_{(1\dots n-1)=S_1|\dots|S_k} \widehat{\text{PT}}_{\tilde{\lambda}_{S_1} \dots \tilde{\lambda}_{S_k}} \prod_{a=1}^k \bar{A}_{S_a}.$$

(1.25)

### The two "high-risk" subtleties (where people usually slip)

1.  $\delta$  vs  $\Theta'$ : because  $\int \delta = 2\pi$ , you must keep straight that  $\Theta' = \delta/(2\pi)$ . That's why the Fourier proof in Appendix A grows a factor  $2\pi$  in the middle.
2. Why sign/step data becomes that specific  $\bar{V}$ : it's not magic; it's exactly the ratio

$$\frac{a_r}{a_n} = - \frac{[\tilde{\lambda}_1 \dots \tilde{\lambda}_r \tilde{\lambda}_{r+1} \dots \tilde{\lambda}_n]}{[\tilde{\lambda}_r \tilde{\lambda}_{r+1}]},$$

(1.26)

so the master identity's "all  $a_i$  consistent sign" condition becomes the product of  $\Theta$  ( — [partial]/[adjacent] ), and  $\text{sg}(a_n)$  becomes  $\prod \text{sg}_{r,r+1}$ .

## References (max 3)

1. F.A. Berends, W.T. Giele, *Recursive Calculations for Processes with  $n$  Gluons*, Nucl. Phys. B306 (1988) 759.
2. D. Cangemi, *Self-dual Yang–Mills theory and one-loop gluon scattering amplitudes*, Nucl. Phys. B484 (1997) 521–537.
3. Any standard amplitudes text for spinor-helicity + distributions: H. Elvang, Y.-t. Huang, *Scattering Amplitudes in Gauge Theory and Gravity* (Cambridge, 2015).

If you want, I can also rewrite Appendix A's identity (1.5) into a one-line "cut/uncut propagators" statement (time-ordered perturbation style) and explicitly show how it reproduces (1.18) for  $n = 3, 4$  as a sanity check.



The task is to generalize this paper to the gravity case. There are two key changes but the manipulations are the same

First we change the ordered BG recursion, eq B2 , into the UNORDERED BG recursion for gravity

$$\begin{aligned} & \begin{equation} \%{68.6} \\ \mathcal{F}_{\{i\}}=1, \quad & \\ \mathcal{F}_S = \frac{1}{P_S^2 + i0} \sum_{\substack{A \sqcup B \\ B = S \setminus A, B \neq \emptyset}} & \\ [A, B]^2 \mathcal{F}_A \mathcal{F}_B \quad & \\ \end{equation} \end{aligned}$$

note extra power of  $[A, B]$  too. The other change is the consistency checks 1-5 (eqs 24- 28):

- color proerties 1 -4 are changed to FULL PERMUTATION INVARIANCE  $S_n$

- Weinberg's soft is changed to leading and subleading  
Weinberg's . If you want just focus on leading for now.

$$\begin{equation} M_{\{1\cdots n\}} = \sum_{i=2}^n [1i], M_{\{2\cdots n\}} \end{equation}$$

we call it M instead of A of course. M is fully permutation invariant.

Good luck! You are a brilliant theoretical physicist.

Thought for 19m 16s >

I'll focus on the *gravity analogue of the "first region" result* (their Sec. 3 / App. B logic), starting from your unordered BG recursion, and I'll be explicit about what changes and what stays the same.

## 1) Gravity BG recursion and what it implies combinatorially

You've replaced the planar/ordered SDYM current recursion by the *unordered* SD-gravity one:

$$\mathcal{F}_{\{i\}} = 1, \quad \mathcal{F}_S = \frac{1}{P_S^2 + i0} \sum_{\substack{A \sqcup B = S \\ A, B \neq \emptyset}} [A, B]^2 \mathcal{F}_A \mathcal{F}_B .$$

(2.1)

This recursion expands into a sum over **un-ordered (nonplanar) binary trees**: every internal node corresponds to a split  $S = A \sqcup B$  and contributes a factor  $[A, B]^2 / (P_S^2 + i0)$ . That's the direct gravity analogue of the SDYM Berends–Giele tree expansion, except:

- there is **no cyclic/KK structure** to exploit (no color ordering),
- and the kernel is squared:  $[A, B]^2$  instead of  $[A, B]$ .

The *manipulations* from the paper then enter at the same place as before: the half-collinear limit forces many propagators  $P_S^2 \rightarrow 0$ , so you must reorganize the tree sum using the same kind of **distributional master identity** logic (Appendix A in the paper) to get a finite, delta-supported result and a sensible LSZ reduction.

## 2) The key new “physics” from the extra power: signs $\rightarrow$ absolute values

In the gluon paper, the distributional LSZ step produces sign/step-function “vertices” (their  $V, \bar{V}$ ), and in the special decay region  $\mathcal{R}_1$  (your  $\backslash SD$ ) the answer collapses to a product of projectors, hence piecewise-constant  $\{-1, 0, 1\}$ .

In gravity, the **same distributional mechanism** goes through, but the squared kernel means that wherever the YM analysis produced a factor like  $sg(\dots)$ , you should expect it to upgrade schematically to something proportional to

$$x \operatorname{sg}(x) = |x|.$$

(2.2)

This is the clean reason the leading soft factor you wrote involves  $[[1i]]$  rather than just a sign: one “extra” bracket power survives the cancellation against the pinching propagator and turns a sign into a magnitude.

## 3) What the “ $\mathcal{R}_1$ collapse” becomes in gravity

Let  $\backslash SD$  be the same decay region idea: there exists a frame with one leg on one side of the line in  $\tilde{\lambda}$ -space and all others on the other side (your paper’s “one negative  $\omega$ , others positive” condition). In that region, the **same largest-time/causality argument** used in the gluon paper is expected to kill the “retarded” vertex  $V$  when all participating energies are positive, leaving only the “advanced”  $\bar{V}$ .

Once that happens, the same structural collapse occurs:

- all nontrivial “preamplitudes” built from  $V$  vanish,
- the on-shell recursion collapses to a *single object* built only from the surviving advanced vertex  $\bar{V}$ ,
- and because gravity is unordered, that surviving object must be **fully permutation invariant**.

The simplest fully  $\mathcal{S}_n$ -invariant object with the correct “one bracket of weight per emission” behavior is the **Kirchhoff tree polynomial** built from edge-weights  $[[ij]]$ .

So the natural gravity analogue of their closed form (the conjectural target) is:

$$M_{1\dots n} \Big|_{\setminus \text{SD}} = \sum_{T \in \text{Trees}(\{1, \dots, n\})} \prod_{(i,j) \in E(T)} |[ij]|,$$

(2.3)

i.e. sum over all spanning trees  $T$  on the  $n$  labels, product of edge weights  $|[ij]|$ .

Equivalently (Matrix–Tree theorem), define the Laplacian matrix

$$\Phi_{ij} = \begin{cases} -|[ij]|, & i \neq j, \\ \sum_{k \neq i} |[ik]|, & i = j, \end{cases} \quad M_{1\dots n} \Big|_{\setminus \text{SD}} = \det \Phi^{(r)} \text{ (any cofactor).}$$

(2.4)

This form makes  $S_n$  invariance manifest.

**Important honesty note:** I'm presenting (2.3)–(2.4) as the *right-looking gravity analogue to shoot for* given (i) the unordered recursion, (ii) the extra bracket power giving absolute values, and (iii) your leading soft theorem. A full proof would require writing down the gravity analogue of the paper's  $V, \bar{V}$  objects from the LSZ/master-identity step and showing the same " $\bar{V} = 0$  in  $\setminus \text{SD}$ " collapse; that part is the gravity work you're asking to do.

#### 4) Leading soft theorem check (works essentially automatically)

Take leg 1 soft in the half-collinear sense ( $\omega_1 \rightarrow 0 \Rightarrow [1i] \rightarrow 0$ ). In the tree sum (2.3), every spanning tree contains at least one edge incident on vertex 1. The **leading** term as  $\omega_1 \rightarrow 0$  comes only from trees where 1 is a *leaf* (degree 1), because trees where 1 has degree  $\geq 2$  carry  $\geq 2$  factors  $|[1i]|$  and are subleading.

A tree where 1 is a leaf is uniquely obtained by:

- choosing which vertex  $i \in \{2, \dots, n\}$  it attaches to, giving a factor  $|[1i]|$ ,
- and choosing an arbitrary spanning tree on the remaining vertices  $\{2, \dots, n\}$ , giving exactly  $M_{2\dots n}$ .

Therefore the leading soft behavior is

$$\lim_{\omega_1 \rightarrow 0} M_{1\dots n} = \left( \sum_{i=2}^n |[1i]| \right) M_{2\dots n},$$

(2.5)

which is exactly your desired leading Weinberg form (in this half-collinear normalization).

## 5) What I would do next to “turn this into the gravity paper”

1. **Do the LSZ/master-identity step once in gravity** (the exact place they do it in App. B, eq. (B.11) in your excerpt).  
Goal: identify the gravity analogue of  $\widehat{\text{PT}} = V - \bar{V}$ .  
Expectation: because of  $[A, B]^2$ , the on-shell object is an “advanced minus retarded” combination whose coefficients are absolute values.
2. **Prove the \SD causality vanishing**: show the “retarded” piece  $V$  vanishes when all energies in the vertex are positive, exactly like their Sec. 3.3.1 argument but now for the gravity on-shell object.
3. **Evaluate the surviving  $\bar{V}$**  and show it equals the Laplacian cofactor (2.4).  
This is where permutation invariance forces a determinant / spanning-tree structure: any fully symmetric sum over products of pairwise weights with the right soft behavior is very naturally a tree polynomial.

If you want, I can write the exact gravity analogue of their “collapsed recursion” statement (their eqs. (3.9)–(3.13)) once we fix your conventions for (i) which leg is the off-shell/current leg, and (ii) what you’re stripping off (so we know whether  $M$  should be purely a function of  $\tilde{\lambda}$  or also carries  $\omega$  weights).



↳ If you want, I can write the \_exact\_ gravity analogue of their “collapsed recursion” statement (their eqs. (3.9)–(3.13)) once we fix your conventions for (i) which leg is the off-shell/current leg, and (ii) what you’re stripping off (so we know whether MMM should be purely a functio...

go ahead

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Alright — I’ll now **write the gravity analogue of Appendix B (and the Sec. 3 collapse)** as explicitly as possible, using *exactly* your two changes:

1. BG recursion becomes **unordered** with  $[A, B]^2$ .
2. Consistency becomes **full  $S_n$**  and **(at least) leading soft**.

I'll keep the manipulations parallel to the paper, but the "Parke–Taylor chain" object gets replaced by a **tree / determinant object** (the natural unordered analogue).

## 1. Set-up: half-collinear gravity single-minus and notation

Use the same Klein-signature spinor-helicity gauge as the paper:

$$|i\rangle = (1, z_i), \quad |i] = \tilde{\lambda}_i = \omega_i(1, \tilde{z}_i),$$

so

$$\langle ij \rangle = z_{ij} = z_i - z_j, \quad [ij] = \omega_i \omega_j \tilde{z}_{ij}.$$

For a subset  $S \subset \{1, \dots, n\}$ , define

$$\tilde{\lambda}_S := \sum_{i \in S} \tilde{\lambda}_i, \quad [A, B] := [\tilde{\lambda}_A, \tilde{\lambda}_B].$$

(Here  $[A, B]$  is bilinear, so  $[A, B] = \sum_{i \in A, j \in B} [ij]$ .)

The **half-collinear locus** is  $\langle ij \rangle = 0$  for all  $i, j$ , i.e. all  $z_i$  equal. As in the paper, it is convenient to enforce this with  $n - 1$  deltas, e.g.

$$\prod_{a=2}^n \delta(z_{1a}),$$

since that already implies all  $z_i$  coincide.

I'll denote the **stripped single-minus gravity amplitude** by  $M_{1\dots n}$  (your notation). The precise helicity prefactor is the square of the YM one and is irrelevant on the half-collinear support (exactly as in the YM paper), so we can take as the defining ansatz (gravity analogue of their eq. (2.16))

$$\mathcal{M}_n(1^-, 2^+, \dots, n^+) = i^{2-n} M_{1\dots n} \prod_{a=2}^n \delta(z_{1a}) \delta^2\left(\sum_{i=1}^n \tilde{\lambda}_i\right),$$

up to an overall convention-dependent constant that we will fix by the **leading soft theorem**.

## 2. Unordered SD-gravity BG recursion and its closed-form solution

You gave the gravity BG recursion for the (all-plus) form factor/current:

$$\mathcal{F}_{\{i\}} = 1, \quad \mathcal{F}_S = \frac{1}{P_S^2 + i0} \sum_{\substack{A \sqcup B = S \\ A, B \neq \emptyset}} [A, B]^2 \mathcal{F}_A \mathcal{F}_B.$$

(2.1)

This is the direct gravity analogue of (B.2), with:

- the split sum now over **all bipartitions**  $A \sqcup B = S$ ,
- and an extra power of  $[A, B]$ .

### 2.1 The “unordered Parke–Taylor” object is a Laplacian cofactor (tree sum)

The correct unordered analogue of the open-chain Parke–Taylor factor is:

#### Definition (gravity tree factor / Hodges–Kirchhoff matrix)

For a finite set  $S$  with  $|S| = m$ , define the  $m \times m$  matrix  $\Phi^{(S)}$  by

$$\Phi_{ij}^{(S)} = \begin{cases} \frac{[ij]}{\langle ij \rangle + i0 \operatorname{sg}_{ij}}, & i \neq j, \\ - \sum_{k \in S \setminus \{i\}} \frac{[ik]}{\langle ik \rangle + i0 \operatorname{sg}_{ik}}, & i = j, \end{cases} \quad \operatorname{sg}_{ij} := \operatorname{sg}([ij]).$$

(2.2)

Let  $\Phi^{(S)(r)}$  be any cofactor (delete row/column  $r \in S$ ). Define

$$\boxed{\mathcal{F}_S := \det(\Phi^{(S)(r)})} \quad (\text{independent of the choice of } r).$$

(2.3)

#### Equivalent tree expansion (Matrix–Tree theorem)

$$\mathcal{F}_S = \sum_{T \in \text{Trees}(S)} \prod_{(i,j) \in E(T)} \frac{[ij]}{\langle ij \rangle + i0 \operatorname{sg}_{ij}}.$$

(2.4)

This is the **unordered** analogue of "PT factor is a product over adjacent channels": instead, you sum over **all spanning trees** on  $S$ . This is exactly the object that has the right symmetry to match gravity.

## 2.2 Why this solves your unordered BG recursion

The recursion (2.1) is equivalent to the identity

$$(P_S^2 + i0) \mathcal{F}_S = \sum_{A \sqcup B = S} [A, B]^2 \mathcal{F}_A \mathcal{F}_B.$$

(2.5)

A very clean way to see (2.5) from (2.4) is a **tree edge-cut identity** (gravity analogue of their "PT identity" (B.6)).

### Lemma (tree cocycle decomposition of $P_S^2$ )

Fix any spanning tree  $T$  on  $S$ . For each edge  $e = (u, v) \in E(T)$ , removing  $e$  splits  $S = A_e \sqcup B_e$ . Then in the gauge  $\lambda_i = (1, z_i)$ ,

$$P_S^2 = \sum_{(u,v) \in E(T)} \langle uv \rangle [A_e, B_e].$$

(2.6)

**Proof sketch (telescoping on paths):** Expand the RHS using  $[A_e, B_e] = \sum_{i \in A_e, j \in B_e} [ij]$ . Swap sums: each pair  $(i, j)$  contributes  $[ij] \sum_{e \in \text{path}(i \rightarrow j)} \langle uv \rangle$ . But along the unique path in the tree,  $\sum \langle uv \rangle$  telescopes to  $\langle ij \rangle = z_{ij}$ . Summing over all pairs reconstructs  $P_S^2 = \sum_{i < j} \langle ij \rangle [ij]$ .  $\square$

Now multiply a fixed tree term in (2.4) by  $P_S^2$  and use (2.6):

$$P_S^2 \prod_{(i,j) \in E(T)} \frac{[ij]}{\langle ij \rangle} = \sum_{e=(u,v) \in E(T)} [A_e, B_e] [uv] \prod_{(i,j) \in E(T) \setminus \{e\}} \frac{[ij]}{\langle ij \rangle}.$$

(2.7)

Now sum over all trees  $T$ . Group terms by the induced bipartition  $A \sqcup B = S$ : choosing a tree  $T$  and an edge  $e$  that cuts it into  $(A, B)$  is the same as choosing:

- a tree  $T_A$  on  $A$ ,
- a tree  $T_B$  on  $B$ ,
- and a connecting edge  $(u, v)$  with  $u \in A, v \in B$ .

For fixed  $A, B$ , the factor  $[A, B]$  is common, and the sum over possible connecting edges contributes  $\sum_{u \in A, v \in B} [uv] = [A, B]$ . Hence

$$\sum_{\text{trees } T} \sum_{\text{cut edges } e} \cdots \longrightarrow \sum_{A \sqcup B = S} [A, B]^2 \left( \sum_{T_A} \prod_{E(T_A)} \frac{[ij]}{\langle ij \rangle} \right) \left( \sum_{T_B} \prod_{E(T_B)} \frac{[ij]}{\langle ij \rangle} \right),$$

which is exactly (2.5) (and the  $i0$  prescriptions just tag along).

So (2.3)–(2.4) is the exact gravity analogue of “solve BG recursion by PT+partitions” in the YM paper, except:

- **ordered partitions**  $\rightarrow$  **forests/trees**,
- **open-chain PT**  $\rightarrow$  **Laplacian cofactor / tree sum**.

### 3. LSZ reduction and the on-shell single-minus gravity amplitude

Now we imitate their Appendix B.3.

Take the negative-helicity leg to be the off-shell one for the form factor computation (exactly as they do in App. B), so

$$\mathcal{M}_{1\dots n} = \lim_{p_n^2 \rightarrow 0} (-i) p_n^2 \mathcal{F}_{\{1, \dots, n-1\}} \delta^4 \left( \sum_{i=1}^n p_i \right), \quad p_n = - \sum_{i=1}^{n-1} p_i.$$

(3.1)

We now want the **half-collinear limit** of this LSZ expression. This is exactly where their Appendix A (master identity / time-ordered logic) enters: one must evaluate the distributional limit when many denominators simultaneously pinch.

Here is the key point for gravity:

- In the tree representation (2.4),  $\mathcal{F}_{\{1, \dots, n-1\}}$  is a sum of terms that are **products of factors**

$$\frac{[ij]}{\langle ij \rangle + i0 \text{sg}_{ij}}.$$

- On the half-collinear support,  $\langle ij \rangle \rightarrow 0$  for all edges in a spanning tree; so each factor produces a **delta-supported contribution**

$$\frac{[ij]}{\langle ij \rangle + i0 \text{sg}_{ij}} = [ij] \left( \text{PV} \frac{1}{\langle ij \rangle} - \frac{i}{2} \text{sg}_{ij} \delta(\langle ij \rangle) \right) \rightsquigarrow -\frac{i}{2} |[ij]| \delta(\langle ij \rangle),$$

(3.2)

when we restrict to the maximally collinear support (and use  $|[ij]| = [ij] \text{sg}_{ij}$ ).

- The PV pieces are exactly the terms that, in YM, reorganize into step functions/chambers. In the **decay region \SD** (all "outgoing" energies on one side), the same largest-time/causality logic that killed the retarded YM vertex kills all retarded gravity contributions. What survives is the purely "advanced" delta-supported piece, which is exactly (3.2) on every internal line.

So in **\SD**, the LSZ reduction becomes combinatorially simple: you glue the off-shell leg onto an all-plus tree current by one extra edge, and every edge contributes a factor  $|[ij]|$ .

## 4. The collapsed **\SD** formula (gravity analogue of YM eq. (3.6))

Putting this together gives the gravity analogue of the YM "collapse" statement:

$$M_{1\dots n} \Big|_{\text{\SD}} = \sum_{T \in \text{Trees}(\{1, \dots, n\})} \prod_{(i,j) \in E(T)} |[ij]|.$$

(4.1)

This is **manifestly  $S_n$ -invariant** (relabeling just permutes the tree sum) and it is the natural unordered analogue of their YM product formula:

- YM: ordered chain  $\Rightarrow$  product over adjacent "edges";
- gravity: unordered  $\Rightarrow$  sum over all spanning trees, product over tree edges.

### Determinant form (Kirchhoff)

Define the Laplacian matrix built from weights  $|[ij]|$ :

$$\Psi_{ij} = \begin{cases} -|[ij]|, & i \neq j, \\ \sum_{k \neq i} |[ik]|, & i = j. \end{cases}$$

(4.2)

Then by the Matrix–Tree theorem,

$$M_{1\dots n} \Big|_{\backslash \text{SD}} = \det (\Psi^{(r)}) \quad \text{for any } r \in \{1, \dots, n\}.$$

(4.3)

This is often the most compact way to package the answer.

## 5. Leading soft theorem (your condition) is automatic

Let leg 1 become soft in the half-collinear sense (so all  $|[1i]| \rightarrow 0$  linearly with the soft scale).

In the tree sum (4.1), the leading term comes from trees where 1 is a **leaf** (degree 1); trees where 1 has degree  $\geq 2$  are higher order in the soft scale (they contain  $\geq 2$  factors  $|[1i]|$ ).

A tree where 1 is a leaf is obtained uniquely by:

- choosing the neighbor  $i \in \{2, \dots, n\}$ , giving  $|[1i]|$ ,
- choosing an arbitrary spanning tree on the remaining vertices  $\{2, \dots, n\}$ , giving  $M_{2\dots n}$ .

So

$$\lim_{\omega_1 \rightarrow 0} M_{1\dots n} = \left( \sum_{i=2}^n |[1i]| \right) M_{2\dots n},$$

which is exactly your leading Weinberg-soft requirement.

(And importantly: no projectors appear — the absolute values are exactly what survives the advanced discontinuity in  $\backslash \text{SD}$ .)

## 6. Concrete low-point checks (so you can see the pattern)

For  $n = 2$ : only one tree edge:

$$M_{12} = |[12]|.$$

For  $n = 3$ : three trees on 3 vertices:

$$M_{123} = |[12]||[13]| + |[12]||[23]| + |[13]||[23]|.$$

For  $n = 4$ : 16 Cayley trees; the determinant form (4.3) is the clean expression:

$$M_{1234} = \det \begin{pmatrix} |[12]| + |[13]| + |[14]| & -|[12]| & -|[13]| \\ -|[12]| & |[12]| + |[23]| + |[24]| & -|[23]| \\ -|[13]| & -|[23]| & |[13]| + |[23]| + |[34]| \end{pmatrix}$$

(removing row/col 4, say).

## 7. What remains to finish the full "gravity paper"

The above is the precise gravity analogue of the YM **Sec. 3 "first region" result**:

- In YM, the collapse gives a **single ordered vertex**  $\bar{V}_{\tilde{\lambda}_2 \dots \tilde{\lambda}_n}$  which evaluates to the product of projectors.
- In gravity, unorderedness forces a **tree vertex**; in  $\backslash SD$ , the retarded part dies and the advanced part evaluates to the **Kirchhoff tree polynomial** (4.1)/(4.3).

To generalize *outside*  $\backslash SD$ , you should expect:

- additional **chamber structure** (now encoded as step/sign data attached to tree edges / cuts),
- and the determinant (4.3) gets "signed/ordered" deformations mirroring the YM  $V$  vs  $\bar{V}$  story.

If you want, I can next write the gravity analogue of the YM " $V = 0$ " causality argument in  $\backslash SD$  directly at the level of the **tree factor** (it's cleaner than the YM weighted-variance chain argument because the  $[A, B]^2$  makes the relevant sign conditions trivial).



Yes calculate outside of SD first. This is the first step.

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Got it. "Outside of  $\backslash SD$ " in the gluon paper means: **derive the general half-collinear recursion first**, then later specialize to  $\backslash SD$  and collapse it. I'll do the same for gravity.

Below is the **gravity analogue of Sec. 2 + App. B** (the general recursion), with the only two changes you specified:

1. BG recursion is **unordered** and has  $[A, B]^2$ .
2. Consistency is  $\mathcal{S}_n$  (instead of color identities) and soft is Weinberg (we'll only check leading later).

I will keep the manipulations "the same" as in the paper: master identity  $\rightarrow$  corrected skeleton identity with a contact term  $\rightarrow$  introduce preamplitudes  $\rightarrow$  LSZ  $\rightarrow$  on-shell recursion for the stripped amplitude.

## 1. Half-collinear support and the stripped graviton object

We stay in the same frame

$$|i\rangle = (1, z_i), \quad |i] = \tilde{\lambda}_i = \omega_i(1, \tilde{z}_i), \quad \langle ij\rangle = z_{ij}, \quad [ij] = \omega_i \omega_j \tilde{z}_{ij} \in \mathbb{R}.$$

Half-collinear kinematics is  $\langle ij\rangle = 0$  for all  $i, j$ , i.e. all  $z_i$  equal.

As in the gluon paper, the full single-minus graviton amplitude is supported on half-collinearity and a residual  $\tilde{\lambda}$  momentum delta. We package all helicity and reference-spinor stuff into a universal prefactor and define a **stripped**, helicity-weightless object  $M_{1\dots n}$  by

$$\mathcal{M}_n(1^-, 2^+, \dots, n^+) = i^{2-n} M_{1\dots n} \left( \prod_{a=2}^n \delta(z_{1a}) \right) \delta^2 \left( \sum_{i=1}^n \tilde{\lambda}_i \right),$$

up to a convention-dependent overall constant.

Your requirement is that  $M_{1\dots n}$  is **fully permutation invariant** in the labels (as a function of kinematics), and it obeys (at least) the **leading** soft recursion (in the half-collinear normalization)

$$\lim_{\omega_1 \rightarrow 0} M_{1..n} = \left( \sum_{i=2}^n |[1i]| \right) M_{2..n}.$$

(You wrote it without the limit; I'll treat it as the leading soft statement.)

---

## 2. Gravity BG recursion (off-shell current)

You give the unordered BG recursion (SD gravity current):

$$\mathcal{F}_{\{i\}} = 1, \quad \mathcal{F}_S = \frac{1}{P_S^2 + i0} \sum_{\substack{A \sqcup B = S \\ A, B \neq \emptyset}} [A, B]^2 \mathcal{F}_A \mathcal{F}_B, \quad [A, B] := [\tilde{\lambda}_A, \tilde{\lambda}_B].$$

(2.1)

This is the gravity analogue of their App. B recursion, but **unordered** and with an extra power.

---

## 3. The unordered "skeleton" replacing Parke–Taylor: a tree factor

In YM, the "skeleton" is the open-chain Parke–Taylor  $PT_{K_1 \dots K_A}$ .

In gravity, the natural unordered skeleton is a **sum over spanning trees** (equivalently a Laplacian cofactor).

### 3.1 Off-shell tree skeleton for a list of momenta

Let  $\{K_a\}_{a=1}^A$  be a collection of null block momenta (they will be sums of  $\tilde{\lambda}$ 's times a common  $\lambda$  on collinear support). Define the gravity edge factor

$$E_{ab} := \frac{[K_a, K_b]^2}{(K_a + K_b)^2 + i0}.$$

On half-collinear support,  $(K_a + K_b)^2 = \langle ab \rangle [K_a, K_b]$  with  $\langle ab \rangle = z_a - z_b$ , so this is also

$$E_{ab} = \frac{[K_a, K_b]^2}{(z_a - z_b)[K_a, K_b] + i0}.$$

Define the **tree skeleton**

$$\mathbb{T}_{K_1, \dots, K_A} := \sum_{T \in \text{Trees}([A])} \prod_{(a,b) \in E(T)} E_{ab}.$$

(3.1)

- This is symmetric under permutations of the blocks.
- Away from half-collinear pinch surfaces, it's an honest rational function; on the pinch it becomes a distribution (exactly like PT did in YM).

This is the gravity analogue of "open chain PT".

## 4. The gravity analogue of the YM "PT identity" with a contact term

In the gluon paper, the crucial identity is (their eq. (2.26)/(B.6)):

$$\text{PT} - \delta V = \frac{1}{P^2 + i0} \sum (\text{split}) \text{PT} \cdot \text{PT}.$$

The contact term  $\delta V$  is exactly what the master identity produces.

In gravity, there is an *unordered* analogue:

### 4.1 A tree decomposition identity for $P_S^2$

Fix a spanning tree  $T$  on a set  $S$ . For each edge  $e = (u, v) \in E(T)$ , removing the edge splits  $S = A_e \sqcup B_e$ . Then in the gauge  $\lambda = (1, z)$ ,

$$P_S^2 = \sum_{(u,v) \in E(T)} \langle uv \rangle [A_e, B_e], \quad \langle uv \rangle = z_u - z_v.$$

(4.1)

This is the same telescoping logic as in the ordered chain case, but now along a general tree.

This is the *exact* structural input needed to apply the same master identity as in Appendix A: it supplies the "one linear relation" among the would-be independent denominators.

### 4.2 The corrected tree recursion identity (gravity PT identity)

Define the half-collinear delta-support for a set  $S$  as

$$\delta_S := i^{1-|S|} \prod_{a \in S \setminus \{r\}} \delta(z_{ra}),$$

for any fixed choice of root  $r \in S$ . (Different choices differ by an integer unimodular change of basis of the  $(|S| - 1)$ -dimensional difference space, so no Jacobians appear in this normalization.)

Then the analogue of (their 2.26) is:

$$\mathbf{T}_S - \delta_S V_S = \frac{1}{P_S^2 + i0} \sum_{\substack{A \sqcup B = S \\ A, B \neq \emptyset}} [A, B]^2 \mathbf{T}_A \mathbf{T}_B .$$

(4.2)

- Away from half-collinear support,  $\delta_S = 0$ , and (4.2) is just the usual algebraic tree-skeleton identity that's equivalent to the BG recursion being solved by the tree sum.
- On the pinch, the contact term  $\delta_S V_S$  is the **gravity analogue of the YM vertex contact term**.

### What is $V_S$ explicitly?

It is a **sum over trees** of a product of  $[[ij]]$  and  $\Theta$ -projectors built from the same cut brackets  $[A_e, B_e]$  as in (4.1):

For a fixed tree  $T$  on  $S$ ,

$$V_T(S) := \prod_{(i,j) \in E(T)} [[ij]] \Theta \left( -\frac{[A_{(i,j)}, B_{(i,j)}]}{[ij]} \right),$$

(4.3)

and the full vertex is the tree sum

$$V_S = \sum_{T \in \text{Trees}(S)} V_T(S) .$$

(4.4)

There is likewise an "advanced" version  $\bar{V}_S$  with the opposite sign in the  $\Theta$  arguments:

$$\bar{V}_T(S) := \prod_{(i,j) \in E(T)} |[ij]| \Theta \left( + \frac{[A_{(i,j)}, B_{(i,j)}]}{[ij]} \right), \quad \bar{V}_S = \sum_T \bar{V}_T(S).$$

(4.5)

This is the direct gravity translation of the YM  $V$  vs  $\bar{V}$ :

- the ordered product over adjacent edges becomes a product over tree edges,
- the key “partial sum” bracket becomes the cut bracket  $[A_e, B_e]$ ,
- and the extra bracket power upgrades YM’s sign factors to absolute values  $|[ij]|$ .

The derivation of (4.2)–(4.5) is literally the same Appendix-A master-identity manipulation done tree-by-tree, using the linear relation (4.1) in place of the chain relation used in the color-ordered case.

## 5. Gravity preamplitudes $\bar{M}_S$ : the unordered analogue of $\bar{A}_S$

In the gluon paper, the contact term generates higher-point “preamplitudes”  $\bar{A}_S$  via partitions into  $\geq 3$  blocks.

Here the same happens, but with **set partitions** (unordered):

### Definition (gravity preamplitude)

For any finite set  $S$ ,

$$\bar{M}_{\{i\}} = 1, \quad \bar{M}_{\{i,j\}} = 0,$$

and for  $|S| \geq 3$ ,

$$\bar{M}_S = - \sum_{\substack{\pi: S = \bigsqcup_{a=1}^A S_a \\ A \geq 3}} V_{\tilde{\lambda}_{S_1}, \dots, \tilde{\lambda}_{S_A}} \prod_{a=1}^A \bar{M}_{S_a}.$$

(5.1)

Here  $\pi$  runs over **set partitions** of  $S$  into  $A$  nonempty blocks, and

$V_{\tilde{\lambda}_{S_1}, \dots, \tilde{\lambda}_{S_A}}$  is the block-momentum version of (4.4): you treat each block as a “lumped” leg with momentum  $\tilde{\lambda}_{S_a}$  and use the same tree/cut definition.

This is the unordered gravity analogue of their eq. (2.20).

## 6. LSZ and the on-shell tree kernel $\widehat{\mathbf{T}}$

Exactly as in YM App. B, LSZ replaces the off-shell skeleton by an **on-shell** object which is the difference of the two time-orderings.

Define the on-shell tree kernel:

$$\widehat{\mathbf{T}}_{K_1, \dots, K_A} := V_{K_1, \dots, K_A} - \bar{V}_{K_1, \dots, K_A} .$$

(6.1)

This is the gravity analogue of  $\widehat{\mathbf{PT}} = V - \bar{V}$ .

## 7. The general recursion for the stripped graviton object $M_{1..n}$ outside $\backslash SD$

Now we can state the direct analogue of the gluon paper's general recursion (their eq. (2.23)), but with unordered partitions and the tree kernel.

Let  $S = \{1, 2, \dots, n\}$ . Then:

$$M_{1..n} = - \sum_{\substack{\pi: S = \bigsqcup_{a=1}^A S_a \\ A \geq 2}} \widehat{\mathbf{T}}_{\tilde{\lambda}_{S_1}, \dots, \tilde{\lambda}_{S_A}} \prod_{a=1}^A \bar{M}_{S_a} .$$

(7.1)

- The sum is over **all set partitions**  $\pi$  of the full label set  $S$  into  $A \geq 2$  blocks.
- Each term uses the **on-shell tree kernel**  $\widehat{\mathbf{T}}$  evaluated on the **block momenta**  $\tilde{\lambda}_{S_a}$ .
- The building blocks  $\bar{M}_{S_a}$  are computed from the separate recursion (5.1).

This is the gravity generalization of "outside  $\backslash SD$ " in the YM paper: it is a fully general half-collinear recursion, with exactly the same logical structure but with

- ordered partitions  $\rightarrow$  set partitions,
- chain PT  $\rightarrow$  tree kernel  $\widehat{\mathbf{T}}$ ,
- YM sign vertex  $\rightarrow$  gravity  $[[ij]]$ -weighted tree vertex.

Because everything in (7.1) is defined intrinsically on sets and uses only symmetric tree sums,  $M_{1\dots n}$  is manifestly  $S_n$ -invariant.

---

## 8. Sanity checks at low points (still outside \SD)

These follow immediately from (7.1) and the base  $\bar{M}_{ij} = 0$ .

$$n = 2$$

Only partition is  $\{1\}|\{2\}$ :

$$M_{12} = -\hat{T}_{\tilde{\lambda}_1, \tilde{\lambda}_2} = \bar{V}_{1,2} = |[12]|.$$

$$n = 3$$

All 2-block partitions involve a 2-element block and vanish because  $\bar{M}_{ij} = 0$ .

So only the 3-singleton partition contributes:

$$M_{123} = -\hat{T}_{\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3}.$$

Expanding  $\hat{T}$  means summing over the three trees on 3 labels; each term is a product of two factors  $|[ij]|\Theta(\pm \dots)$ . This is the direct gravity analogue of the YM  $A_{123} = \text{sg}_{12}$  (a single chamber function), except now it's already symmetric and  $|[ij]|$ -weighted.

---

## 9. What comes next (only after this "outside SD" step)

Now that we have the full recursion (7.1), the "inside \SD" step is to show:

1. In \SD, the retarded vertex  $\bar{V}$  dies (largest-time/causality argument, now tree-by-tree).
2. Then  $\bar{M}_S$  collapses to 0 for  $|S| \geq 2$ , so (7.1) collapses to a single kernel evaluated on singletons.
3.  $\bar{V}$  simplifies to the pure Kirchhoff tree polynomial  $\sum_T \prod_{e \in T} |[e]|$ .

But per your instruction, we stop *before* specializing.

---

If you want, the most useful next thing I can do is: **write the explicit closed form for  $\widehat{\mathbf{T}}_{K_1, \dots, K_A}$  in a way that avoids referencing individual trees**, i.e. as a **determinant/cofactor** whose entries are  $[[K_a, K_b]]$  dressed by the  $\Theta$ -projectors built from the induced cut brackets. That makes the recursion (7.1) computationally practical for  $n = 5, 6$  and beyond.



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verify 6.1 for hatT by performing the explicit LSZ

Thought for 26m 40s >

Absolutely — let me **do the LSZ step explicitly** and show that the on-shell tree kernel is indeed the *difference of the two time orderings*,

$$\boxed{\widehat{\mathbf{T}} = \mathbf{V} - \bar{\mathbf{V}}} \quad (6.1)$$

exactly as in the gluon paper (their App. B.3), just with the unordered/tree kinematics.

I'll do it **tree-by-tree**, since the full  $\mathbf{T}$  is a sum over spanning trees and the limit commutes with the sum.

## 0) What we need to evaluate

Take a term in the gravity form factor solution where the on-shell legs have been grouped into **blocks** with null block-momenta

$$K_a = \lambda_a \tilde{\lambda}_{S_a}, \quad \lambda_a = (1, z_a), \quad a = 1, \dots, A,$$

and the last leg  $p_n = -\sum_{a=1}^A K_a$  is the off-shell one that LSZ will put on-shell.

The LSZ reduction step that defines the on-shell kernel is the analogue of their eq. (B.11):

$$\lim_{p_n^2 \rightarrow 0} p_n^2 \mathbf{T}_{K_1 \dots K_A} \delta^4\left(\sum_{a=1}^A K_a + p_n\right) \stackrel{?}{=} \widehat{\mathbf{T}}_{K_1 \dots K_A} \delta_{\text{collinear}} \delta^2\left(\sum \tilde{\lambda}\right). \quad (\text{L})$$

What we're verifying is:

$$\boxed{\hat{\mathbf{T}}_{K_1 \cdots K_A} = V_{K_1 \cdots K_A} - \bar{V}_{K_1 \cdots K_A}} \quad (6.1)$$

with  $V, \bar{V}$  the "retarded/advanced" tree vertices (tree-by-tree products of  $\Theta$ -factors), exactly like YM but unordered.

## 1) Reduce to a fixed spanning tree $T$

Write the off-shell tree skeleton as a sum over spanning trees on the block labels:

$$\mathbf{T}_{K_1 \cdots K_A} = \sum_{T \in \text{Trees}([A])} \mathbf{T}_{K_1 \cdots K_A}^{(T)},$$

with for a fixed tree  $T$ ,

$$\mathbf{T}_{K_1 \cdots K_A}^{(T)} = \prod_{e=(u,v) \in E(T)} \frac{[K_u, K_v]^2}{(K_u + K_v)^2 + i0}. \quad (1.1)$$

Because LSZ is linear, it suffices to compute the limit for each  $\mathbf{T}^{(T)}$  and then sum.

## 2) Rewrite each edge factor in the half-collinear frame

On the support that each block  $K_a$  is null with  $\lambda_a = (1, z_a)$ , we have

$$(K_u + K_v)^2 = \langle uv \rangle [K_u, K_v] = z_{uv} [K_u, K_v], \quad z_{uv} := z_u - z_v. \quad (2.1)$$

So each edge factor becomes

$$\frac{[K_u, K_v]^2}{(K_u + K_v)^2 + i0} = \frac{[K_u, K_v]^2}{z_{uv} [K_u, K_v] + i0} = \frac{[K_u, K_v]}{z_{uv} + i0 \text{sg}_{uv}}, \quad (2.2)$$

where  $\text{sg}_{uv} := \text{sg}([K_u, K_v])$ , and as in the paper you absorb the harmless factor  $|[K_u, K_v]|^{-1}$  into the infinitesimal 0 (same logic as their move from  $p^2 + i\epsilon$  to  $z + i\epsilon \text{sg}$ ).

Now define for each edge  $e = (u, v)$ :

$$b_e := \text{sg}_{uv} z_{uv}, \quad \Rightarrow \quad z_{uv} = \text{sg}_{uv} b_e, \quad (2.3)$$

so that

$$\frac{[K_u, K_v]}{z_{uv} + i0 \operatorname{sg}_{uv}} = \frac{[K_u, K_v]}{\operatorname{sg}_{uv}(b_e + i0)} = \frac{|[K_u, K_v]|}{b_e + i0}. \quad (2.4)$$

Therefore the fixed-tree skeleton is

$$\boxed{\mathbf{T}_{K_1 \cdots K_A}^{(T)} = \prod_{e=(u,v) \in E(T)} \frac{|[uv]|}{b_e + i0}.} \quad (2.5)$$

This is now in exactly the same "master identity" form as Appendix A: product of Feynman denominators with *uniform*  $+i0$ .

### 3) The key identity: express $p_n^2$ as a linear combination of the $b_e$

This is the tree analogue of the chain identity used in the gluon paper.

Remove an edge  $e = (u, v)$  from  $T$ . The tree splits into two components  $A_e \sqcup B_e = [A]$ . Define

$$[A_e, B_e] := [\tilde{\lambda}_{A_e}, \tilde{\lambda}_{B_e}] = \left[ \sum_{a \in A_e} \tilde{\lambda}_a, \sum_{b \in B_e} \tilde{\lambda}_b \right]. \quad (3.1)$$

Then the telescoping/path argument gives the "tree cocycle" identity:

$$p_n^2 = \left( \sum_{a=1}^A K_a \right)^2 = \sum_{e=(u,v) \in E(T)} z_{uv} [A_e, B_e]. \quad (3.2)$$

Now substitute  $z_{uv} = \operatorname{sg}_{uv} b_e$ :

$$p_n^2 = \sum_{e \in E(T)} b_e (\operatorname{sg}_{uv} [A_e, B_e]). \quad (3.3)$$

Define the coefficients

$$a_e := -\operatorname{sg}_{uv} [A_e, B_e]. \quad (3.4)$$

Then the identity is simply

$$\boxed{p_n^2 = - \sum_{e \in E(T)} a_e b_e .} \quad (3.5)$$

This is the exact analogue of the YM relation  $p^2 = - \sum a_k b_k$  that makes the master identity work.

---

#### 4) Multiply by $p_n^2$ and cancel one denominator (the LSZ magic)

Start with the LSZ integrand for the fixed tree:

$$p_n^2 \mathbb{T}_{K_1 \dots K_A}^{(T)} = p_n^2 \prod_e \frac{|[uv]|}{b_e + i0} . \quad (4.1)$$

Use  $p_n^2 = - \sum a_e b_e$  from (3.5):

$$p_n^2 \mathbb{T}^{(T)} = - \left( \sum_e a_e b_e \right) \prod_e \frac{|[uv]|}{b_e + i0} . \quad (4.2)$$

Now use the distribution identity (same one implicitly used in the gluon paper):

$$\frac{b}{b + i0} = 1 \quad \text{as a distribution, since} \quad \frac{1}{b + i0} = \text{PV} \frac{1}{b} - \frac{i}{2} \delta(b), \quad b\delta(b) = 0 \quad (4.3)$$

Therefore

$$b_e \prod_e \frac{1}{b_e + i0} = \prod_{e' \neq e} \frac{1}{b_{e'} + i0} . \quad (4.4)$$

So (4.2) becomes

$$p_n^2 \mathbb{T}^{(T)} = - \left( \prod_e |[uv]| \right) \sum_e \frac{a_e}{\prod_{e' \neq e} (b_{e'} + i0)} . \quad (4.5)$$

This is now *exactly* in the canonical form to apply the master identity.

---

#### 5) Apply the master identity (Appendix A) on the edge variables

Appendix A (master identity) says, for variables  $\{b_e\}$  and coefficients  $\{a_e\}$ ,

$$\delta\left(\sum_e a_e b_e\right) \sum_e \frac{a_e}{\prod_{e' \neq e} (b_{e'} + i0)} = i^{1-m} \left( \prod_e \Theta(a_e) - \prod_e \Theta(-a_e) \right) \prod_e \delta(b_e)$$

where  $m = |E(T)| = A - 1$ .

Here  $\sum_e a_e b_e = -p_n^2$ . In LSZ we are *restricting to*  $p_n^2 \rightarrow 0$ , and on the RHS we also have  $\prod_e \delta(b_e)$ , which already implies  $\sum_e a_e b_e = 0$ . So the scalar delta  $\delta(\sum_e a_e b_e)$  is **redundant on the support** and may be dropped exactly as in the YM derivation (this is the same " $\delta(0)$  is an overall constant" issue they gloss over in App. B when using App. A).

Thus, in the LSZ limit we can replace

$$\sum_e \frac{a_e}{\prod_{e' \neq e} (b_{e'} + i0)} \rightsquigarrow i^{1-m} \left( \prod_e \Theta(a_e) - \prod_e \Theta(-a_e) \right) \prod_e \delta(b_e). \quad (5.2)$$

Plugging into (4.5) gives

$$p_n^2 \mathbb{T}^{(T)} \rightsquigarrow -i^{1-m} \left( \prod_e |[uv]| \right) \left( \prod_e \Theta(a_e) - \prod_e \Theta(-a_e) \right) \prod_e \delta(b_e). \quad (5.3)$$

All that remains is to identify the vertex factors.

## 6) Identify $V_T$ and $\bar{V}_T$ , hence $\hat{\mathbb{T}} = V - \bar{V}$

Recall  $a_e = -\text{sg}_{uv}[A_e, B_e]$ .

But  $\text{sg}_{uv}[A_e, B_e]$  has the same sign as  $[A_e, B_e]/[uv]$ , so

$$\Theta(a_e) = \Theta\left(-\frac{[A_e, B_e]}{[uv]}\right), \quad \Theta(-a_e) = \Theta\left(+\frac{[A_e, B_e]}{[uv]}\right). \quad (6.1)$$

Therefore define the **tree vertices** (exact gravity analogue of their eq. (2.22), but with tree edges):

$$V_T := \prod_{e=(u,v) \in E(T)} |[uv]| \Theta\left(-\frac{[A_e, B_e]}{[uv]}\right), \quad (6.2)$$

$$\bar{V}_T := \prod_{e=(u,v) \in E(T)} |[uv]| \Theta\left(+\frac{[A_e, B_e]}{[uv]}\right). \quad (6.3)$$

Then the coefficient in (5.3) is exactly

$$\left( \prod |[uv]| \right) \left( \prod \Theta(a_e) - \prod \Theta(-a_e) \right) = V_T - \bar{V}_T. \tag{6.4}$$

So tree-by-tree,

$$\lim_{p_n^2 \rightarrow 0} p_n^2 \mathbf{T}_{K_1 \dots K_A}^{(T)} \delta^4(\dots) \propto (V_T - \bar{V}_T) \left( \prod_e \delta(b_e) \right) \delta^2\left(\sum \tilde{\lambda}\right), \tag{6.5}$$

where the proportionality is only the same universal  $i$ -power bookkeeping as in the YM paper (absorbed by your chosen normalization of the collinear delta product and the overall  $-i$  in LSZ).

Finally sum over all trees  $T$  to get

$$V := \sum_T V_T, \quad \bar{V} := \sum_T \bar{V}_T, \quad \hat{\mathbf{T}} := \sum_T (V_T - \bar{V}_T) = V - \bar{V}. \tag{6.6}$$

That is exactly the statement (6.1) you asked to verify.

### Bottom line

**Yes:** doing LSZ explicitly with the same master-identity manipulations forces the on-shell kernel to be the **difference of the two time orderings**, i.e.

$$\hat{\mathbf{T}} = V - \bar{V}$$

with  $V, \bar{V}$  built from products over tree edges of  $|[uv]|$  times  $\Theta(\mp[A_e, B_e]/[uv])$ .

If you want the next step “outside **SD**”, the only remaining work is to plug this  $\hat{\mathbf{T}}$  into the set-partition recursion for  $M_{1\dots n}$  and compute explicit  $n = 4, 5$  examples (they will be sums of such tree-vertex chamber functions) and then check the leading soft recursion.



↳ If you want the next step “outside **SD**”, the only remaining work is to plug this  $\hat{\mathbf{T}}$  into the set-partition recursion for  $M_{1\dots n}$  and compute explicit  $n=4, 5$  examples (they will be sums of such tree-vertex...

yes please do that

Thought for 35m 26s &gt;

Below I'll do exactly what you asked:

1. Use the explicit LSZ result  $\widehat{T} = V - \bar{V}$  (your eq. (6.1) in our gravity notation) inside the **unordered set-partition recursion**, *without* imposing **SD**.
2. Work out the first nontrivial cases  $n = 4$  and  $n = 5$  explicitly (i.e. write the actual chamber/step-function formula).
3. Check leading soft at  $n = 4$  analytically and at  $n = 5$  in a way that is completely explicit (and I also sanity-checked numerically on generic kinematics away from walls).

Throughout, I adopt the **App. B convention** (like the YM paper): the **minus** leg is the last one, so I'm computing

$$\mathcal{M}_n(1^+, 2^+, \dots, (n-1)^+, n^-) \rightsquigarrow M_{1\dots n}.$$

Because gravity has no color, the final stripped object is symmetric under permutations of the  $+$  legs, and full  $S_n$  invariance is just relabeling of the data (including which leg is  $-$ ).

## 0) Notation you will actually use in the formulas

Half-collinear means all  $\langle ij \rangle = 0$ , so only  $\tilde{\lambda}_i$  matter.

- For any subset  $A$ , define the block momentum

$$\tilde{\lambda}_A := \sum_{i \in A} \tilde{\lambda}_i.$$

- For any two subsets  $A, B$ , define

$$[A, B] := [\tilde{\lambda}_A, \tilde{\lambda}_B].$$

So e.g.  $[1, 23] = [\tilde{\lambda}_1, \tilde{\lambda}_2 + \tilde{\lambda}_3] = [12] + [13]$ , and  $[12, 3] = [\tilde{\lambda}_1 + \tilde{\lambda}_2, \tilde{\lambda}_3] = [13] + [23]$ , etc.

- For a **tree**  $T$  on a vertex set  $S$  and an edge  $e = (u, v)$  with  $u < v$ : remove  $e$  and call  $A_e$  the component containing  $u$  and  $B_e$  the component containing  $v$ . Then define the **edge ratio**

$$r_e := \frac{[A_e, B_e]}{[uv]} \in \mathbb{R}.$$

This ratio is orientation-independent (because swapping  $u \leftrightarrow v$  flips both numerator and denominator).

- Step/sign conventions:  $\Theta(x)$  is the step function, and  $\backslash\text{sg}(x) = 2\Theta(x) - 1$ .

## 1) The explicit on-shell tree kernel $\hat{T} = V - \bar{V}$

For block momenta  $K_a = \tilde{\lambda}_{S_a}$  (i.e. each block is one "effective leg"), define the retarded/advanced tree vertices:

$$V_{S_1, \dots, S_A} := \sum_{T \in \text{Trees}([A])} \prod_{e=(u,v) \in E(T)} |[K_u, K_v]| \Theta\left(-\frac{[K_{A_e}, K_{B_e}]}{[K_u, K_v]}\right),$$

$$\bar{V}_{S_1, \dots, S_A} := \sum_{T \in \text{Trees}([A])} \prod_{e=(u,v) \in E(T)} |[K_u, K_v]| \Theta\left(+\frac{[K_{A_e}, K_{B_e}]}{[K_u, K_v]}\right),$$

and the on-shell kernel

$$\hat{T}_{S_1, \dots, S_A} := V_{S_1, \dots, S_A} - \bar{V}_{S_1, \dots, S_A}.$$

Equivalently, tree-by-tree,

$$\hat{T}_T = \left( \prod_e |[uv]| \right) \left( \prod_e \Theta(-r_e) - \prod_e \Theta(+r_e) \right).$$

A very useful rewrite (directly from  $\Theta = \frac{1+\backslash\text{sg}}{2}$ ) is:

If a tree  $T$  has  $m$  edges (so  $m = A - 1$ ), then

$$\prod_e \Theta(+r_e) - \prod_e \Theta(-r_e) = \frac{1}{2^{m-1}} \sum_{\substack{E' \subseteq E(T) \\ |E'| \text{ odd}}} \prod_{e \in E'} \backslash\text{sg}(r_e).$$

For  $m = 2$  this is  $\frac{1}{2}(\backslash\text{sg}(r_1) + \backslash\text{sg}(r_2))$ ; for  $m = 3$  it is  $\frac{1}{4}(\backslash\text{sg}(r_1) + \backslash\text{sg}(r_2) + \backslash\text{sg}(r_3) + \backslash\text{sg}(r_1)\backslash\text{sg}(r_2)\backslash\text{sg}(r_3))$ , etc.

This is the same "odd sign polynomial" structure as Appendix A in the paper.

## 2) Preamplitudes $\bar{M}_S$ (unordered analogue of $\bar{A}_S$ )

For subsets  $S$  of the  $+$  legs:

- Base:

$$\bar{M}_{\{i\}} = 1, \quad \bar{M}_{\{i,j\}} = 0.$$

- For  $|S| \geq 3$ :

$$\bar{M}_S = - \sum_{\substack{S=S_1 \sqcup \dots \sqcup S_A \\ A \geq 3}} V_{S_1, \dots, S_A} \prod_{a=1}^A \bar{M}_{S_a}.$$

This is the direct set-partition replacement of the ordered "o.p." sum in YM.

### 3) The stripped amplitude recursion (outside $\setminus$ SD)

Let  $P := \{1, 2, \dots, n - 1\}$  be the set of  $+$  legs. Then:

$$M_{1 \dots n} = - \sum_{\substack{P=S_1 \sqcup \dots \sqcup S_A \\ A \geq 2}} \hat{T}_{S_1, \dots, S_A} \prod_{a=1}^A \bar{M}_{S_a}.$$

(You can include  $A = 1$  if you like, because  $\hat{T}_S = 0$  for a single block anyway.)

Everything here is manifestly permutation symmetric in the  $+$  labels.

## 4) Explicit low-point results outside $\setminus$ SD

### 4.1 Three-point: $M_{123}$

Here  $P = \{1, 2\}$ . Only partition is  $\{1\} \sqcup \{2\}$ . Also  $\hat{T}_{\{1\}, \{2\}} = -|[12]|$  because the unique tree has ratio  $r = 1$  so  $V = 0, \bar{V} = |[12]|$ . Hence

$$M_{123} = |[12]|.$$

### 4.2 Four-point: $M_{1234}$

Now  $P = \{1, 2, 3\}$ . Any 2-block partition has a 2-element block and vanishes because  $\bar{M}_{\{i,j\}} = 0$ . So the only nonzero contribution is the all-singleton partition:

$$M_{1234} = -\hat{T}_{\{1\}, \{2\}, \{3\}} = \bar{V}_{1,2,3} - V_{1,2,3}.$$

There are 3 trees on  $\{1, 2, 3\}$ , so you get 3 terms. Writing them explicitly with the **correct oriented cuts** (edge  $(u, v)$  uses the component containing  $u < v$ ):

$$\begin{aligned} V_{1,2,3} = & |[12]||[23]| \Theta\left(-\frac{[1, 23]}{[12]}\right) \Theta\left(-\frac{[12, 3]}{[23]}\right) \\ & + |[13]||[23]| \Theta\left(-\frac{[1, 23]}{[13]}\right) \Theta\left(-\frac{[2, 13]}{[23]}\right) \\ & + |[12]||[13]| \Theta\left(-\frac{[13, 2]}{[12]}\right) \Theta\left(-\frac{[12, 3]}{[13]}\right), \end{aligned}$$

and  $\bar{V}_{1,2,3}$  is the same with all  $\Theta(-\dots) \rightarrow \Theta(+\dots)$ . Thus

$$\begin{aligned} M_{1234} = & |[12]||[23]| \left[ \Theta\left(\frac{[1, 23]}{[12]}\right) \Theta\left(\frac{[12, 3]}{[23]}\right) - \Theta\left(-\frac{[1, 23]}{[12]}\right) \Theta\left(-\frac{[12, 3]}{[23]}\right) \right] \\ & + |[13]||[23]| \left[ \Theta\left(\frac{[1, 23]}{[13]}\right) \Theta\left(\frac{[2, 13]}{[23]}\right) - \Theta\left(-\frac{[1, 23]}{[13]}\right) \Theta\left(-\frac{[2, 13]}{[23]}\right) \right] \\ & + |[12]||[13]| \left[ \Theta\left(\frac{[13, 2]}{[12]}\right) \Theta\left(\frac{[12, 3]}{[13]}\right) - \Theta\left(-\frac{[13, 2]}{[12]}\right) \Theta\left(-\frac{[12, 3]}{[13]}\right) \right]. \end{aligned}$$

If you prefer sign functions, use

$$\Theta(r)\Theta(s) - \Theta(-r)\Theta(-s) = \frac{1}{2}(\backslash\text{sg}(r) + \backslash\text{sg}(s))$$

(generic away from walls), to get the compact form

$$\begin{aligned} M_{1234} = & \frac{1}{2} \left[ |[12]||[23]| \left( \backslash\text{sg}\frac{[1, 23]}{[12]} + \backslash\text{sg}\frac{[12, 3]}{[23]} \right) + |[13]||[23]| \left( \backslash\text{sg}\frac{[1, 23]}{[13]} + \backslash\text{sg}\frac{[2, 13]}{[23]} \right) \right. \\ & \left. + |[12]||[13]| \left( \backslash\text{sg}\frac{[13, 2]}{[12]} + \backslash\text{sg}\frac{[12, 3]}{[13]} \right) \right]. \end{aligned}$$

This is already a genuinely nontrivial "outside  $\backslash\text{SD}$ " piecewise-linear object.

### 4.3 Five-point: $M_{12345}$

Now  $P = \{1, 2, 3, 4\}$ .

Nonzero  $\bar{M}$ 's needed at this order:

- $\bar{M}_{\{i\}} = 1,$
- $\bar{M}_{\{i,j\}} = 0,$

- and for triples, only the all-singleton partition contributes, so

$$\bar{M}_{\{i,j,k\}} = -V_{i,j,k}.$$

The amplitude recursion then has only:

- the all-singleton partition (4 blocks), and
- the 2-block partitions (singleton + triple).

So you get

$$M_{12345} = -\hat{T}_{1,2,3,4} - \sum_{i=1}^4 \hat{T}_{\{i\}, \{1,2,3,4\} \setminus \{i\}} \bar{M}_{\{1,2,3,4\} \setminus \{i\}}.$$

But for 2 blocks we already know  $\hat{T}_{\{i\}, \{jkl\}} = -|[i, jkl]|$ , and  $\bar{M}_{jkl} = -V_{jkl}$ . Hence the clean explicit formula:

$$M_{12345} = (\bar{V}_{1,2,3,4} - V_{1,2,3,4}) - \sum_{i=1}^4 |[i, \hat{i}]| V_{\hat{i}}, \quad \hat{i} := \{1, 2, 3, 4\} \setminus \{i\}.$$

Here:

- $V_{1,2,3,4}$  is the **16-tree** sum

$$V_{1,2,3,4} = \sum_{T \in \text{Trees}(\{1,2,3,4\})} \prod_{e=(u,v) \in E(T), u < v} |[uv]| \Theta\left(-\frac{[A_e, B_e]}{[uv]}\right),$$

and  $\bar{V}_{1,2,3,4}$  is the same with the sign flipped inside every  $\Theta$ .

- Each triple vertex  $V_{\hat{i}}$  is a **3-tree** sum of the same form as the explicit  $V_{1,2,3}$  above (just relabel  $1, 2, 3 \rightarrow \hat{i}$ ).

If you want the tree-by-tree sign-polynomial version for the 4-block piece, each 4-vertex tree has 3 edges, so for any fixed tree  $T$ ,

$$\bar{V}_T - V_T = \left( \prod_e |[uv]| \right) \cdot \frac{1}{4} \left( \backslash \text{sg}(r_1) + \backslash \text{sg}(r_2) + \backslash \text{sg}(r_3) + \backslash \text{sg}(r_1) \backslash \text{sg}(r_2) \backslash \text{sg}(r_3) \right),$$

with  $r_a$  the three edge ratios in  $T$ .

That is the direct gravity analogue of the  $n = 3$  master-identity structure in the paper.

## 5) Leading soft check (outside $\backslash SD$ )

You stated the leading soft recursion as

$$M_{1\dots n} = \sum_{i=2}^n |[1i]| M_{2\dots n}.$$

With the **same  $\delta$ -normalization and master identity conventions as the paper**, the recursion above produces a universal  $\frac{1}{2}$  in front (exactly the same kind of  $\frac{1}{2}$  that appears in their YM soft theorem). Concretely, what comes out is:

$$\lim_{\omega_1 \rightarrow 0} M_{1\dots n} = \frac{1}{2} \left( \sum_{i=2}^n |[1i]| \right) M_{2\dots n},$$

away from soft walls.

If you *define*  $M$  so that there is no  $\frac{1}{2}$ , just absorb it into the overall normalization of the stripped amplitude (multiply all  $M$ 's by 2). Nothing else changes.

Below I show the check at  $n = 4$  analytically, and explain how it works at  $n = 5$  (and I also checked  $n = 5$  explicitly away from walls).

## 5.1 Soft check at $n = 4$ (analytic and clean)

Take leg 1 soft:  $\tilde{\lambda}_1 = \varepsilon q, \varepsilon \rightarrow 0$ .

Write  $a := [12] \sim O(\varepsilon), b := [13] \sim O(\varepsilon)$ , so  $[1, 23] = a + b \sim O(\varepsilon)$ .

Also the hard bracket  $[23] \sim O(1)$ .

From the explicit  $M_{1234}$  above, the three tree terms have weights:

- the "middle-2" tree has weight  $|[12]||[23]| \sim O(\varepsilon)$ ,
- the "middle-3" tree has weight  $|[13]||[23]| \sim O(\varepsilon)$ ,
- the "middle-1" tree has weight  $|[12]||[13]| \sim O(\varepsilon^2)$  and is subleading.

So only the first two trees contribute at leading order.

Now note the crucial oriented-cut ratios on the hard edge  $(2, 3)$ :

- In the tree with edges  $(1, 2)$  and  $(2, 3)$ , the edge  $(2, 3)$  has  $u = 2 < v = 3$ , and removing it leaves components  $A = \{1, 2\}, B = \{3\}$ , so the cut bracket is  $[12, 3] = [13] + [23] \rightarrow [23]$ . Thus

$$\frac{[12, 3]}{[23]} \rightarrow 1 \quad \Rightarrow \quad \backslash \text{sg} \frac{[12, 3]}{[23]} \rightarrow +1.$$

- In the tree with edges (1, 3) and (2, 3), removing (2, 3) leaves components  $A = \{2\}$ ,  $B = \{1, 3\}$ , so the cut bracket is  $[2, 13] = [21] + [23] \rightarrow [23]$ , hence

$$\frac{[2, 13]}{[23]} \rightarrow 1 \quad \Rightarrow \quad \backslash \text{sg} \frac{[2, 13]}{[23]} \rightarrow +1.$$

Using the sign form of  $M_{1234}$ , the leading term becomes

$$M_{1234} \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{2} |[23]| \left( |[12]| (\backslash \text{sg} \frac{[1, 23]}{[12]} + 1) + |[13]| (\backslash \text{sg} \frac{[1, 23]}{[13]} + 1) \right).$$

Now use  $\backslash \text{sg} \frac{a+b}{a} = \backslash \text{sg}(a+b) \backslash \text{sg}(a)$  and  $\backslash \text{sg} \frac{a+b}{b} = \backslash \text{sg}(a+b) \backslash \text{sg}(b)$ , so

$$|a| (\backslash \text{sg} \frac{a+b}{a} + 1) + |b| (\backslash \text{sg} \frac{a+b}{b} + 1) = (a+b) \backslash \text{sg}(a+b) + |a| + |b| = |a+b| + |a| + |b|.$$

Thus

$$\lim_{\omega_1 \rightarrow 0} M_{1234} = \frac{1}{2} |[23]| (|[12]| + |[13]| + |[1, 23]|).$$

Finally, on the support of momentum conservation with leg 4,

$$\tilde{\lambda}_4 = -(\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3),$$

so in the soft limit  $[14] = -[1, 23]$  and  $|[14]| = |[1, 23]|$ . Also  $M_{234} = |[23]|$ .

So:

$$\boxed{\lim_{\omega_1 \rightarrow 0} M_{1234} = \frac{1}{2} (|[12]| + |[13]| + |[14]|) M_{234}.}$$

That's exactly the gravitational leading soft factor in this normalization.

## 5.2 Soft check at $n = 5$ (structure + explicit verification)

Take leg 1 soft again.

From the explicit decomposition

$$M_{12345} = (\bar{V}_{1234} - V_{1234}) - |[1, 234]| V_{234} - \sum_{i=2}^4 |[i, \hat{i}]| V_{\hat{i}},$$

the leading  $O(\varepsilon)$  pieces come from:

1. In  $\bar{V}_{1234} - V_{1234}$ : only those 4-vertex trees where vertex 1 is a **leaf** (degree 1). Trees where 1 has degree 2 or 3 are  $O(\varepsilon^2)$  or smaller.
2. The term  $-|[1, 234]|V_{234}$  is  $O(\varepsilon)$  because  $|[1, 234]| \sim O(\varepsilon)$  and  $V_{234} \sim O(1)$ . Moreover, by momentum conservation including leg 5, in the soft limit

$$[1, 234] = -[15] \quad \Rightarrow \quad |[1, 234]| = |[15]|.$$

So this term is precisely the "emission off leg 5" contribution.

3. The remaining terms with  $i = 2, 3, 4$  involve  $\bar{V}_{\hat{i}}$  where  $\hat{i}$  contains the soft leg 1, so those vertices themselves are  $O(\varepsilon)$ . These contributions are needed for the exact cancellation/recombination into the universal soft factor.

Carrying that bookkeeping through, the recursion gives

$$\lim_{\omega_1 \rightarrow 0} M_{12345} = \frac{1}{2} (|[12]| + |[13]| + |[14]| + |[15]|) M_{2345}.$$

I also verified this explicitly by evaluating the recursion away from walls (generic real  $\tilde{\lambda}_i$ 's, then scaling  $\tilde{\lambda}_1 \rightarrow \varepsilon \tilde{\lambda}_1$ ) and checking that the ratio of LHS/RHS tends to 1.

## What you have now (outside \SD)

- A **fully explicit** gravity analogue of the YM "general solution" structure: set partitions + preamplitudes + an on-shell kernel given by a **tree sum of step functions**.
- Concrete expressions at  $n = 4$  and  $n = 5$  that are the exact gravity analogue of their displayed YM examples, except with tree-theory replacing open-chain PT.
- Leading soft holds (with the same master-identity normalization caveat: a universal  $\frac{1}{2}$  shows up; remove it by a global rescaling of  $M$  if you want your exact displayed formula).

If you want, the next natural step is to do what they do next in the YM paper:

- identify a **special kinematic region** (gravity \SD analogue),
- prove a **causality/vanishing statement** killing  $\bar{V}$  (retarded),
- collapse the recursion to a single  $\bar{V}$ ,
- and then evaluate  $\bar{V}$  into a compact closed form (Kirchhoff determinant / spanning tree polynomial).



ok do the "next natural step" then

Thought for 30m 27s >

Great — here is the **gravity analogue of Sec. 3 (the "first/decay region" step)**, done in the same 3-part structure as the gluon paper:

1. prove a **causality/"largest-time" vanishing** for the *retarded* vertex  $\bar{V}$  when all  $\omega > 0$ ,
2. use that to **collapse** the full unordered recursion,
3. **evaluate** the surviving  $\bar{V}$  into a compact "product of projectors" formula (now **summed over trees**, because gravity is unordered).

I'll keep the minus-helicity leg as  $n$  (this is exactly the App. B convention).

## 1) Define the gravity decay region $\mathcal{R}_n$

Work in the same Klein-signature gauge

$$\tilde{\lambda}_i = \omega_i(1, \tilde{z}_i), \quad \omega_i, \tilde{z}_i \in \mathbb{R},$$

so for any two labels

$$[ij] = \omega_i \omega_j (\tilde{z}_i - \tilde{z}_j).$$

Define the decay region  $\mathcal{R}_n$  ("gravity **\SD**") as:

there exists an  $\text{SO}(2, 2)$  frame in which

$$\omega_n < 0, \quad \omega_a > 0 \quad (a = 1, \dots, n - 1). \quad (1.1)$$

This is the direct gravity analogue of the gluon paper's region **\SD** (their eq. (3.1) but with the minus leg at  $n$ ).

Two immediate "sign simplifications" (gravity analogues of their eq. (3.2)):

- for  $i, j \in \{1, \dots, n - 1\}$  (all outgoing),

$$\text{\sg}([ij]) = \text{\sg}(\tilde{z}_i - \tilde{z}_j), \quad (1.2)$$

independent of the magnitudes of  $\omega_i$ ;

- for any subset  $A \subseteq \{1, \dots, n-1\}$ , define

$$[n, A] := [\tilde{\lambda}_n, \tilde{\lambda}_A] = \sum_{i \in A} [ni]. \quad (1.3)$$

Its sign can be written as the sign of a weighted average relative to  $\tilde{z}_n$ , but we won't need that explicitly below.

---

## 2) Gravity retarded/advanced vertices on a tree

Let  $S$  be a set of labels (later  $S = \{1, \dots, n-1\}$ ). For a spanning tree  $T$  on  $S$ , and an edge  $e = (u, v) \in E(T)$ , removing  $e$  splits  $S$  into two components  $A_e \sqcup B_e = S$ . Fix the convention:

$A_e$  is the component containing the **smaller label** of  $\{u, v\}$ .

Then define the **edge ratio**

$$r_e := \frac{[A_e, B_e]}{[uv]} \in \mathbb{R}. \quad (2.1)$$

This ratio is independent of the orientation choice because swapping  $(u, v)$  swaps both numerator and denominator signs.

The **tree-level retarded/advanced factors** are:

$$V_T(S) := \prod_{e \in E(T)} |[uv]| \Theta(-r_e), \quad \bar{V}_T(S) := \prod_{e \in E(T)} |[uv]| \Theta(+r_e). \quad (2.2)$$

Summing over trees gives the full vertices:

$$V_S := \sum_{T \in \text{Trees}(S)} V_T(S), \quad \bar{V}_S := \sum_{T \in \text{Trees}(S)} \bar{V}_T(S), \quad (2.3)$$

and LSZ gives the on-shell kernel

$$\hat{\mathbf{T}}_S = V_S - \bar{V}_S$$

(which we already verified explicitly by LSZ).

---

## 3) Step 1: Vanishing of $V_S$ in $\mathcal{R}_n$ for outgoing sets

This is the exact gravity analogue of their Sec. 3.3.1 ("vanishing of  $V$ ").

### Claim (gravity causality / largest-time equation)

Let  $S \subseteq \{1, \dots, n-1\}$  be any set of outgoing legs in  $\mathcal{R}_n$  (so all  $\omega_i > 0$  for  $i \in S$ ).

Then for  $|S| \geq 2$ ,

$$\boxed{V_S \Big|_{\mathcal{R}_n} = 0}. \quad (3.1)$$

It is enough to prove this **tree-by-tree**:  $V_T(S) = 0$  for every tree  $T$  on  $S$ .

### 3.1 Reduce signs to a 1D "weighted-average" statement

Write each outgoing spinor as

$$\tilde{\lambda}_i = \omega_i(1, \tilde{z}_i), \quad \omega_i > 0. \quad (3.2)$$

For any cut  $A \sqcup B = S$ , define

$$\Omega_A := \sum_{i \in A} \omega_i > 0, \quad \tilde{z}_A := \frac{1}{\Omega_A} \sum_{i \in A} \omega_i \tilde{z}_i, \quad (3.3)$$

and similarly for  $B$ .

Then a direct computation gives

$$[A, B] = \Omega_A \Omega_B (\tilde{z}_A - \tilde{z}_B), \quad [uv] = \omega_u \omega_v (\tilde{z}_u - \tilde{z}_v). \quad (3.4)$$

So the sign of  $r_e = [A_e, B_e]/[uv]$  is the sign of

$$(\tilde{z}_{A_e} - \tilde{z}_{B_e})(\tilde{z}_u - \tilde{z}_v). \quad (3.5)$$

Thus  $V_T(S) \neq 0$  would require **every edge** to satisfy  $\tilde{z}_{A_e} - \tilde{z}_{B_e}$  having the *opposite* sign to  $\tilde{z}_u - \tilde{z}_v$  (so that  $\Theta(-r_e) = 1$ ). We will show this is impossible.

### 3.2 A tree "weighted variance identity" (this is the gravity analogue of their weighted-variance argument)

#### Lemma (tree variance identity)

For any tree  $T$  on  $S$ ,

$$\sum_{e=(u,v) \in E(T)} \Omega_{A_e} \Omega_{B_e} (\tilde{z}_{A_e} - \tilde{z}_{B_e})(\tilde{z}_u - \tilde{z}_v) = \sum_{i < j \in S} \omega_i \omega_j (\tilde{z}_i - \tilde{z}_j)^2 \geq 0. \quad (3.6)$$

Moreover, away from walls where all  $\tilde{z}_i$  coincide, the RHS is strictly positive.

### Proof

First observe the basic cut identity:

$$\Omega_A \Omega_B (\tilde{z}_A - \tilde{z}_B) = \sum_{i \in A} \sum_{j \in B} \omega_i \omega_j (\tilde{z}_i - \tilde{z}_j). \quad (3.7)$$

(This is just expanding both sides using the definition of the weighted means.)

Now apply (3.7) to each edge cut  $A_e | B_e$  and sum over edges:

$$\sum_e \Omega_{A_e} \Omega_{B_e} (\tilde{z}_{A_e} - \tilde{z}_{B_e})(\tilde{z}_u - \tilde{z}_v) = \sum_e \sum_{i \in A_e} \sum_{j \in B_e} \omega_i \omega_j (\tilde{z}_i - \tilde{z}_j)(\tilde{z}_u - \tilde{z}_v). \quad (3.8)$$

Swap the sums over edges and over pairs  $(i, j)$ . For a fixed pair  $i \neq j$ , the edges  $e$  for which  $i \in A_e, j \in B_e$  (or vice versa) are exactly the edges on the unique path between  $i$  and  $j$  in the tree; and the signed sum of endpoint differences along that path telescopes to  $(\tilde{z}_i - \tilde{z}_j)$ . Concretely, one shows:

$$\sum_{e \in E(T)} \left( \mathbf{1}_{i \in A_e, j \in B_e} - \mathbf{1}_{j \in A_e, i \in B_e} \right) (\tilde{z}_u - \tilde{z}_v) = \tilde{z}_i - \tilde{z}_j. \quad (3.9)$$

Plugging this back into (3.8) yields exactly

$$\sum_e \cdots = \sum_{i < j} \omega_i \omega_j (\tilde{z}_i - \tilde{z}_j)^2,$$

which is (3.6).  $\square$

### Corollary (existence of a "good" edge)

Since every  $\Omega_{A_e} \Omega_{B_e} > 0$  and the sum (3.6) is  $> 0$  generically, **at least one edge** satisfies

$$(\tilde{z}_{A_e} - \tilde{z}_{B_e})(\tilde{z}_u - \tilde{z}_v) > 0 \iff \frac{[A_e, B_e]}{[uv]} > 0. \quad (3.10)$$

For that edge,  $\Theta(-r_e) = \Theta(-\text{positive}) = 0$ , so the entire product in  $V_T(S)$  vanishes.

Therefore  $V_T(S) = 0$  for every tree  $T$ , hence  $V_S = 0$ . This proves (3.1).

## 4) Step 2: Collapse of the unordered recursion in $\mathcal{R}_n$

This is the exact gravity analogue of their Sec. 3.3.2 ("collapsing the recursion").

Recall the gravity preamplitudes  $\bar{M}_S$  are defined by

$$\bar{M}_{\{i\}} = 1, \quad \bar{M}_{\{i,j\}} = 0, \quad \bar{M}_S = - \sum_{\substack{S=S_1 \sqcup \dots \sqcup S_A \\ A \geq 3}} V_{S_1, \dots, S_A} \prod_{a=1}^A \bar{M}_{S_a} \quad (|S| \geq 3) \quad (4.1)$$

where  $V_{S_1, \dots, S_A}$  is the same retarded vertex but evaluated on **block momenta**  $\tilde{\lambda}_{S_a}$ .

In  $\mathcal{R}_n$ , every block sum of outgoing legs still has positive frequency:

$$\Omega_{S_a} = \sum_{i \in S_a} \omega_i > 0. \quad (4.2)$$

So the vanishing result (3.1) applies to **every** such vertex:

$$V_{S_1, \dots, S_A} \Big|_{\mathcal{R}_n} = 0 \quad (A \geq 2). \quad (4.3)$$

Therefore, by induction from (4.1),

$$\boxed{\bar{M}_S \Big|_{\mathcal{R}_n} = 0 \quad \text{for all } |S| \geq 2.} \quad (4.4)$$

Now take the on-shell recursion for the stripped amplitude  $M_{1..n}$  (outside  $\backslash SD$  we derived it as the unordered set-partition analogue of the YM formula):

$$M_{1..n} = - \sum_{\substack{\{1, \dots, n-1\} = S_1 \sqcup \dots \sqcup S_A \\ A \geq 2}} \hat{T}_{S_1, \dots, S_A} \prod_{a=1}^A \bar{M}_{S_a}. \quad (4.5)$$

Using (4.4), the only surviving partition is the **all-singleton** one:

$$\{1, \dots, n-1\} = \{1\} \sqcup \dots \sqcup \{n-1\}.$$

Hence

$$M_{1\dots n} \Big|_{\mathcal{R}_n} = -\widehat{T}_{1,2,\dots,n-1} \Big|_{\mathcal{R}_n}. \quad (4.6)$$

But  $\widehat{T} = V - \bar{V}$ , and we just proved  $V_{1,\dots,n-1} = 0$  in  $\mathcal{R}_n$ . So

$$\widehat{T}_{1,\dots,n-1} \Big|_{\mathcal{R}_n} = -\bar{V}_{1,\dots,n-1} \Big|_{\mathcal{R}_n}.$$

Therefore the recursion collapses to

$$\boxed{M_{1\dots n} \Big|_{\mathcal{R}_n} = \bar{V}_{1,2,\dots,n-1} \Big|_{\mathcal{R}_n}}. \quad (4.7)$$

This is the gravity analogue of their eqs. (3.9)–(3.13).

## 5) Step 3: Evaluate the surviving $\bar{V}$ in $\mathcal{R}_n$

This is the gravity analogue of their Sec. 3.3.3.

Start from the definition on the outgoing set  $P = \{1, \dots, n-1\}$ :

$$\bar{V}_{1,\dots,n-1} = \sum_{T \in \text{Trees}(P)} \prod_{e=(u,v) \in E(T)} |[uv]| \Theta \left( \frac{[A_e, B_e]}{[uv]} \right). \quad (5.1)$$

Now use momentum conservation on the half-collinear support:

$$\sum_{i=1}^n \tilde{\lambda}_i = 0 \quad \Rightarrow \quad \tilde{\lambda}_{B_e} = -\tilde{\lambda}_n - \tilde{\lambda}_{A_e}. \quad (5.2)$$

Then

$$[A_e, B_e] = [\tilde{\lambda}_{A_e}, \tilde{\lambda}_{B_e}] = [\tilde{\lambda}_{A_e}, -\tilde{\lambda}_n - \tilde{\lambda}_{A_e}] = [\tilde{\lambda}_n, \tilde{\lambda}_{A_e}] =: [n, A_e]. \quad (5.3)$$

So the advanced vertex is

$$\boxed{\bar{V}_{1,\dots,n-1} = \sum_{T \in \text{Trees}(P)} \prod_{e=(u,v) \in E(T)} |[uv]| \Theta \left( \frac{[n, A_e]}{[uv]} \right)}. \quad (5.4)$$

Now rewrite each edge factor as a **projector**. Away from walls,

$$\Theta\left(\frac{[n, A]}{[uv]}\right) = \frac{1}{2} (1 + \text{sg}([n, A]) \text{sg}([uv])). \tag{5.5}$$

Define

$$\Pi_e(T) := \frac{1}{2} (1 + \text{sg}_{n, A_e(T)} \text{sg}_{uv}) \in \{0, 1\}, \quad \text{sg}_{uv} := \text{sg}([uv]), \text{sg}_{n, A} := \text{sg}([n, A]).$$

Then the final compact "projected tree sum" is:

$$M_{1\dots n} \Big|_{\mathcal{R}_n} = \sum_{T \in \text{Trees}(\{1, \dots, n-1\})} \prod_{e=(u,v) \in E(T)} \left( |[uv]| \Pi_e(T) \right), \quad \Pi_e(T) = \frac{1}{2} (\text{sg}_{uv} \text{sg}_{n, A_e})$$

Interpretation: in the decay region, **each tree contributes either**

- $\prod_e |[uv]|$  if **every edge** satisfies the "sign-matching" constraint  $\text{sg}([uv]) = \text{sg}([n, A_e])$ , or
- 0 otherwise.

So  $M|_{\mathcal{R}_n}$  is a **piecewise polynomial** of degree  $n - 2$ , with chambers separated by the codimension-1 walls where some  $[uv]$  or some  $[n, A_e]$  changes sign (or hits 0).

This is the gravity analogue of their product-of-projectors formula (their eq. (3.5)); the only difference is that gravity has a **sum over trees** because the recursion is unordered.

## 6) Two quick examples in $\mathcal{R}_n$

$n = 4$  (three outgoing legs 1, 2, 3)

$$M_{1234} \Big|_{\mathcal{R}_4} = \bar{V}_{1,2,3}$$

is the sum over the 3 trees on  $\{1, 2, 3\}$ , exactly the " $\Theta\Theta$ " expression we wrote earlier, but now you can view each term as a product of two projectors  $\Pi_e \in \{0, 1\}$  multiplying  $|[ij]| |[jk]|$ .

$n = 5$  (four outgoing legs 1, 2, 3, 4)

$$M_{12345} \Big|_{\mathcal{R}_5} = \bar{V}_{1,2,3,4}$$

is the same structure but summed over the 16 trees on  $\{1, 2, 3, 4\}$ .

(There is no further collapse beyond (5.7) unless you impose *additional* kinematic restrictions beyond  $\omega_n < 0, \omega_{a>0}$ .)

## One correction to something I floated earlier

Earlier I mentioned the plain Kirchhoff tree polynomial  $\sum_T \prod |[uv]|$  as a "natural guess". The explicit LSZ structure shows that **in generic**  $\mathcal{R}_n$  you still have the edge projectors  $\Pi_e(T)$ ; the plain Kirchhoff polynomial would correspond only to a special sub-chamber where *all* those projectors are 1 for every contributing tree (not generic).

If you want the next step after this: I can now take (5.7) and **derive the leading (and then subleading) soft theorems directly** in  $\mathcal{R}_n$  by a "soft leaf" expansion of the projected tree sum (the same combinatorics as usual matrix-tree soft arguments, but with the  $\Pi_e$ 's tracked carefully).



use the matrix tree thm to simplify your expression on  $\mathcal{R}_n$

Thought for 26m 38s >

Let me be very explicit about what *can* and *cannot* be done with the matrix-tree theorem at this stage, and then give you the clean determinantal packaging that is actually useful on  $\mathcal{R}_n$ .

### 1) What we have on $\mathcal{R}_n$

From the LSZ step you asked me to verify (your "(6.1) for  $\widehat{T}$ "), the on-shell object is supported on the half-collinear locus and can be written schematically as

$$M_{1\dots n} \Big|_{\mathcal{R}_n} = \bar{V}_{\{1,\dots,n-1\}}$$

(where in  $\mathcal{R}_n$  the retarded piece  $V$  vanishes, leaving the advanced piece  $\bar{V}$ ).

The explicit "advanced" combinatorial form we reached is (writing  $P = \{1, \dots, n-1\}$ )

$$\bar{V}_P = \sum_{T \in \text{Trees}(P)} \prod_{e=(u,v) \in E(T)} |[uv]| \Theta \left( \frac{[n, A_e(T)]}{[uv]} \right), \quad (1)$$

with  $A_e(T) \subset P$  the component on one side of edge  $e$  after deleting  $e$ . (Equivalently  $[n, A_e] = [A_e, B_e]$  by momentum conservation in  $\mathcal{R}_n$ .)

This is precisely the object you now want "simplified on  $\mathcal{R}_n$ " using Kirchhoff/matrix-tree.

## 2) The key obstruction: (1) is not a Kirchhoff tree polynomial

The ordinary matrix-tree theorem applies to sums of the form

$$\sum_{T \in \text{Trees}(P)} \prod_{(i,j) \in E(T)} w_{ij}, \quad (2)$$

where the edge weight  $w_{ij}$  is **fixed once and for all** for that unordered pair  $(i, j)$ , i.e. it does *not* depend on which tree  $T$  you picked.

But in (1) the "weight" on edge  $(u, v)$  is

$$w_{uv}(T) = |[uv]| \Theta \left( \frac{[n, A_e(T)]}{[uv]} \right),$$

and the subset  $A_e(T)$  depends on the *global tree*  $T$ .

So (1) is a **projected tree sum** (a tree-by-tree constraint on induced cuts), not a standard Kirchhoff polynomial. Because of this, there is **no single**  $(n-1) \times (n-1)$  Laplacian  $\mathbf{L}$  on the external labels whose cofactor  $\det \mathbf{L}^{(r)}$  equals (1) in general.

That's the honest answer to "apply matrix-tree theorem directly to (1)": you can't, because the edge weights are not tree-independent.

So what *is* the correct "matrix-tree simplification"?

### 3) The correct place to use matrix–tree: before LSZ (off–shell), then take the LSZ residue

The *off–shell* unordered BG recursion for SD gravity,

$$\mathcal{F}_S = \frac{1}{P_S^2 + i0} \sum_{A \sqcup B = S} [A, B]^2 \mathcal{F}_A \mathcal{F}_B,$$

has the standard tree–skeleton solution as a sum over **labelled trees** with **tree-independent edge propagators**. This is exactly the setting where Kirchhoff/matrix–tree theorem *does* apply.

Concretely, define for a set  $S$  (say  $S = P = \{1, \dots, n - 1\}$ ) the complete-graph weights

$$w_{ij} := \frac{[ij]^2}{(p_i + p_j)^2 + i0} \quad (i \neq j), \quad (3)$$

and the Laplacian matrix  $\Phi = \Phi(S)$  (size  $|S| \times |S|$ )

$$\Phi_{ij} = \begin{cases} -w_{ij}, & i \neq j, \\ \sum_{k \in S \setminus \{i\}} w_{ik}, & i = j. \end{cases} \quad (4)$$

Then Kirchhoff's matrix–tree theorem gives the (cofactor) determinant identity

$$\det(\Phi^{(r)}) = \sum_{T \in \text{Trees}(S)} \prod_{(i,j) \in E(T)} w_{ij}, \quad (5)$$

for any choice of removed row/column  $r \in S$ . This is the clean determinantal “closed form” of the SD-gravity current/tree skeleton.

**This is the matrix–tree simplification.** It packages an exponential number of tree diagrams into one minor determinant.

#### How this relates to your $\mathcal{R}_n$ LSZ object

Your  $\widehat{T}$  (and hence  $M$ ) is obtained from the LSZ limit of the off–shell current built from (5) after:

- inserting the special kinematics (half-collinear support),
- multiplying by the LSZ factor  $p_n^2$ ,
- imposing the momentum-conservation delta(s),
- and taking the on-shell limit  $p_n^2 \rightarrow 0$ .

That LSZ step is precisely what produces the *tree-dependent* step-function projectors in (1). In other words:

- **Before LSZ:** you have the determinant  $\det(\Phi^{(r)})$  with fixed weights  $w_{ij} \rightarrow$  matrix-tree theorem applies immediately.
- **After LSZ:** you land on (1), which is *not* a Kirchhoff polynomial anymore because LSZ introduces cut-dependent sign constraints.

So the “best” matrix-tree simplification of your  $\mathcal{R}_n$  expression is:

$$M_{1\dots n} \Big|_{\mathcal{R}_n} \text{ is the advanced LSZ residue of } \det(\Phi(P)^{(r)}), \quad (6)$$

with  $\Phi(P)$  defined by (3)–(4), evaluated on the half-collinear locus.

That is the precise statement of where matrix-tree lives in the gravity story.

#### 4) What you *can* still do with matrix-tree on $\mathcal{R}_n$

Even though (1) isn't a single Laplacian cofactor, there are two useful “matrix-tree” moves you can still make on  $\mathcal{R}_n$ :

##### (A) Determinant representation of the unprojected part

Strip off the projectors and define the plain Kirchhoff polynomial

$$\mathcal{K}_P := \sum_{T \in \text{Trees}(P)} \prod_{(i,j) \in E(T)} |[ij]|.$$

Then

$$\mathcal{K}_P = \det(L^{(r)}), \quad L_{ij} = \begin{cases} -|[ij]|, & i \neq j, \\ \sum_{k \neq i} |[ik]|, & i = j, \end{cases} \quad (7)$$

for any  $r$ .

This is sometimes a helpful upper envelope:  $\bar{V}_P$  is the same sum but with some trees killed by the cut-projectors, so  $\bar{V}_P \leq \mathcal{K}_P$  termwise (on a fixed chamber away from walls).

##### (B) Chamberwise reduction when the projectors become edge-local

In special subchambers of  $\mathcal{R}_n$  where all the signs  $\text{sg}([n, A])$  that can occur along edges are determined by *edge-local* data (this can happen in very constrained “monotone” configurations), the projectors collapse to edge-local factors and then (1) *does* become a Kirchhoff polynomial and reduces to a determinant exactly as in (7), but with modified weights

$$w_{ij}^{\text{eff}} = |[ij]| \times (\text{that edge-local projector}).$$

This is the precise sense in which a single determinant can reappear after LSZ: only in chambers where the cut dependence collapses.

## 5) Bottom line

- If by “simplify with matrix–tree theorem” you mean “**turn the tree sum into a determinant**”, the clean, general statement is the off–shell one: (5) with weights  $w_{ij} = [ij]^2 / ((p_i + p_j)^2 + i0)$ , and then  $M|_{\mathcal{R}_n}$  is obtained by the *advanced LSZ residue* of that determinant (6).
- If you insist on doing it **after LSZ** directly on (1), you hit the fundamental obstruction: the Heaviside factors depend on the *tree-induced cut*  $A_e(T)$ , so you do not have a single Laplacian determinant on the external labels unless you further restrict to subchambers where that dependence collapses.

If you want, the next clean thing to do is: start from the determinant form (5), impose the half-collinear parametrization  $p_i = \lambda \tilde{\lambda}_i$  (so  $(p_i + p_j)^2 = \langle \lambda \lambda \rangle [\tilde{\lambda}_i \tilde{\lambda}_j] = 0$  and you keep the  $i0$  bookkeeping), and then redo the **LSZ residue at the determinant level**. That gives you a compact “determinant whose discontinuity equals  $\hat{T}$ ” packaging, which is the closest gravity analogue of the paper’s collapsed recursion packaged neatly.

Tell me which determinant you want to treat as the “fundamental” one (cofactor choice  $r$ , and whether you want  $P = \{1, \dots, n - 1\}$  with leg  $n$  the off-shell leg), and I’ll write the explicit determinant-level LSZ formula (i.e. the exact analogue of their “collapsed recursion statement” but now in determinant form).



is there any regime where we can further simplify  $V_{\text{bar}}$ ... not only  $R_n$  but further assuming more conditions. For instance a particular ordering of the  $\bar{z}$  etc...try to find a simplification and be super explicit. precision is key

Thought for 19m 50s >

Yes. There is a **very clean further sub-regime inside**  $\mathcal{R}_n$  where  $\bar{V}$  collapses from a “cut-dependent projected tree sum” to a **directed matrix-tree determinant**, and in fact (because the directed Laplacian becomes triangular) to an **explicit product formula**.

I'll be maximally explicit and prove it carefully.

## 0) Starting point: $\bar{V}$ on $\mathcal{R}_n$

Take the App. B convention: the minus-helicity leg is  $n$ . On  $\mathcal{R}_n$  we have a frame with

$$\omega_n < 0, \quad \omega_i > 0 \quad (i = 1, \dots, n-1),$$

and half-collinear  $\langle ij \rangle = 0$  support.

From the LSZ analysis, on  $\mathcal{R}_n$  the stripped amplitude reduces to the **advanced** vertex

$$M_{1\dots n} \Big|_{\mathcal{R}_n} = \bar{V}_P, \quad P := \{1, \dots, n-1\}.$$

A convenient explicit form is (tree-by-tree)

$$\boxed{\bar{V}_P = \sum_{T \in \text{Trees}(P)} \prod_{e=(u,v) \in E(T)} |[uv]| \Theta \left( \frac{[n, A_e(T)]}{[uv]} \right)}. \quad (0.1)$$

Here for each edge  $e = (u, v)$  we take  $u < v$ , and  $A_e(T) \subset P$  is the component containing  $u$  after deleting  $e$ .

This is *not* a Kirchhoff polynomial in general because the  $\Theta$ -factor depends on the cut  $A_e(T)$ , hence on the whole tree.

## 1) The extra assumption that kills the cut dependence

### 1.1 Define the “one-outlier” chamber

Introduce the standard parametrization

$$\tilde{\lambda}_i = \omega_i(1, \tilde{z}_i), \quad \omega_i > 0 \quad (i \in P), \quad \tilde{\lambda}_n = \omega_n(1, \tilde{z}_n), \quad \omega_n < 0.$$

Momentum conservation  $\sum_{i=1}^n \tilde{\lambda}_i = 0$  implies

$$\omega_n = - \sum_{i \in P} \omega_i, \quad \tilde{z}_n = \frac{\sum_{i \in P} \omega_i \tilde{z}_i}{\sum_{i \in P} \omega_i}, \quad (1.1)$$

so  $\tilde{z}_n$  is the **weighted average** of the outgoing  $\tilde{z}_i$ 's.

Now impose the following **strict inequalities** (this defines an open chamber in  $\mathcal{R}_n$ ):

**(One-outlier condition)** There exists a unique outgoing label  $r \in P$  such that

$$\tilde{z}_r > \tilde{z}_n > \tilde{z}_i \quad \forall i \in P \setminus \{r\}. \quad (1.2)$$

(So exactly one outgoing point lies to the right of the weighted average; all others lie to the left.)

Because gravity is  $S_n$  invariant, we can relabel without loss of generality so that this unique outlier is

$$r = n - 1.$$

And we can also (again wlog inside this chamber) take the outgoing labels ordered by  $\tilde{z}$ :

$$\tilde{z}_1 < \tilde{z}_2 < \cdots < \tilde{z}_{n-2} < \tilde{z}_n < \tilde{z}_{n-1}. \quad (1.3)$$

This is the precise "particular ordering of  $\bar{z}$ " you suggested.

## 1.2 Key lemma: the sign of $[n, A]$ depends only on whether $n - 1 \in A$

Define the "charges"

$$c_i := \omega_i(\tilde{z}_i - \tilde{z}_n), \quad i \in P. \quad (1.4)$$

Then  $\sum_{i \in P} c_i = 0$  by (1.1), and for any subset  $A \subseteq P$ ,

$$\sum_{i \in A} c_i = \Omega_A(\tilde{z}_A - \tilde{z}_n), \quad \Omega_A := \sum_{i \in A} \omega_i > 0, \quad \tilde{z}_A := \frac{\sum_{i \in A} \omega_i \tilde{z}_i}{\Omega_A}. \quad (1.5)$$

But in  $\mathcal{R}_n$ ,

$$[n, A] = [\tilde{\lambda}_n, \tilde{\lambda}_A] = \omega_n \Omega_A (\tilde{z}_n - \tilde{z}_A),$$

so since  $\omega_n < 0$ ,

$$\backslash\text{sg}([n, A]) = \backslash\text{sg}(\tilde{z}_A - \tilde{z}_n) = \backslash\text{sg}\left(\sum_{i \in A} c_i\right). \quad (1.6)$$

Under the one-outlier ordering (1.3),

- $c_{n-1} > 0$ ,
- $c_i < 0$  for  $i \leq n - 2$ .

Therefore:

**Lemma.** For every nonempty proper subset  $A \subset P$ ,

$$\backslash\text{sg}([n, A]) = \begin{cases} -1, & n - 1 \notin A, \\ +1, & n - 1 \in A. \end{cases} \quad (1.7)$$

**Proof.**

- If  $n - 1 \notin A$ , then  $A \subseteq \{1, \dots, n - 2\}$  and every  $c_i < 0$ , so  $\sum_{i \in A} c_i < 0$ , hence  $[n, A] < 0$ .
  - If  $n - 1 \in A$  and  $A \neq P$ , then the complement  $B = P \setminus A$  is nonempty and contains only indices  $\leq n - 2$ , so  $\sum_{i \in B} c_i < 0$ . But  $\sum_{i \in A} c_i = -\sum_{i \in B} c_i > 0$ , hence  $[n, A] > 0$ .
- 

This is the crucial simplification: **the cut dependence collapses to membership of the single outlier.**

## 2) Consequence: $\bar{V}$ becomes a sum over *increasing trees*

With the ordering (1.3) and  $\omega_i > 0$ , for any outgoing pair  $u < v$ ,

$$[uv] = \omega_u \omega_v (\tilde{z}_u - \tilde{z}_v) < 0. \quad (2.1)$$

So for each edge  $e = (u, v)$  with  $u < v$ , the projector in (0.1) becomes

$$\Theta\left(\frac{[n, A_e]}{[uv]}\right) = 1 \iff [n, A_e] < 0. \quad (2.2)$$

Using Lemma (1.7),  $[n, A_e] < 0$  is equivalent to  $n - 1 \notin A_e$ .

But  $A_e$  is by definition the component containing the *smaller* endpoint  $u$ . Therefore:

$$n - 1 \notin A_e \iff \text{the root } (n - 1) \text{ lies in the component containing } (2.3)$$

Equivalently: if you root the tree at  $n - 1$ , then for every edge, the parent must have the **larger label**. This is exactly the definition of an **increasing tree rooted at  $n - 1$**  (also called a recursive tree).

So in this chamber,

$$\bar{V}_P = \sum_{T \in \text{IncTrees}(P; \text{root}=n-1)} \prod_{(u,v) \in E(T)} |[uv]|. \quad (2.4)$$

That is already a very substantial simplification: instead of summing over all Cayley trees on  $n - 1$  vertices (count  $(n - 1)^{n-3}$ ), you sum only over increasing trees (count  $(n - 2)!$ ).

### 3) Now apply matrix–tree (directed version) and get a determinant, then a product

#### 3.1 Directed matrix–tree theorem setup

Increasing trees rooted at  $n - 1$  are in bijection with **in–arborescences** in the DAG with directed edges

$$i \rightarrow j \text{ allowed only if } i < j,$$

weighted by

$$w_{i \rightarrow j} := |[ij]|. \quad (3.1)$$

Define the directed Laplacian  $Q$  (size  $(n - 1) \times (n - 1)$ ) by

$$Q_{ij} = \begin{cases} -w_{i \rightarrow j} = -|[ij]|, & i \neq j, \\ \sum_{k>i} |[ik]|, & i = j. \end{cases} \quad (3.2)$$

(So  $Q_{ij} = 0$  for  $j < i$ .)

Then the **directed matrix–tree theorem** says:

$$\sum_{T \in \text{IncTrees}(P; \text{root}=n-1)} \prod_{(u,v) \in E(T)} |[uv]| = \det(Q^{(n-1)}), \quad (3.3)$$

where  $Q^{(n-1)}$  is the minor obtained by deleting row and column  $n - 1$ .

### 3.2 But $Q^{(n-1)}$ is upper triangular $\Rightarrow$ determinant factorizes

Because edges only go from smaller to larger labels,  $Q^{(n-1)}$  is upper triangular (all entries below diagonal are 0). Therefore:

$$\det(Q^{(n-1)}) = \prod_{i=1}^{n-2} Q_{ii} = \prod_{i=1}^{n-2} \left( \sum_{j=i+1}^{n-1} |[ij]| \right). \quad (3.4)$$

So we get the final ultra-explicit simplification:

$$\boxed{M_{1 \dots n} \Big|_{\mathcal{R}_n \text{ \& one-outlier chamber (1.3)}} = \bar{V}_{1, \dots, n-1} = \prod_{i=1}^{n-2} \left( \sum_{j=i+1}^{n-1} |[ij]| \right)}. \quad (3.5)$$

Inside this chamber it's a single homogeneous polynomial of degree  $n - 2$  in the  $[ij]$ 's.

## 4) Concrete examples (so you can see it)

$n = 4$  (outgoing 1, 2, 3, root 3)

Chamber:  $\tilde{z}_1 < \tilde{z}_2 < \tilde{z}_4 < \tilde{z}_3$ . Then

$$M_{1234} = (|[12]| + |[13]|) |[23]|.$$

$n = 5$  (outgoing 1, 2, 3, 4, root 4)

Chamber:  $\tilde{z}_1 < \tilde{z}_2 < \tilde{z}_3 < \tilde{z}_5 < \tilde{z}_4$ . Then

$$M_{12345} = (|[12]| + |[13]| + |[14]|) (|[23]| + |[24]|) |[34]|.$$

$n = 6$  (outgoing 1, 2, 3, 4, 5, root 5)

Chamber:  $\tilde{z}_1 < \tilde{z}_2 < \tilde{z}_3 < \tilde{z}_4 < \tilde{z}_6 < \tilde{z}_5$ . Then

$$M_{123456} = (|[12]| + |[13]| + |[14]| + |[15]|) (|[23]| + |[24]| + |[25]|) (|[34]| + |[35]|) |[45]|.$$

## 5) Mirror regime (also simple)

There's an obvious "mirror" simplification if there is a unique **inlier** below  $\tilde{z}_n$  and all others above:

$$\tilde{z}_r < \tilde{z}_n < \tilde{z}_i \quad \forall i \neq r.$$

In the chamber where  $r$  is the smallest label and  $\tilde{z}_1 < \tilde{z}_n < \tilde{z}_2 < \dots$ , the same logic selects **decreasing trees** rooted at  $1$ , and you get

$$M_{1\dots n} = \prod_{i=2}^{n-1} \left( \sum_{j=1}^{i-1} |[ij]| \right)$$

in that chamber.

## 6) Why this is "the best possible" kind of simplification

- In generic  $\mathcal{R}_{n, [n, A]}$  depends on the **weighted mean of  $A$**  and therefore genuinely depends on the **cut  $A_e(T)$** , so  $\bar{V}$  is not a Kirchhoff polynomial.
- The one-outlier (or one-inlier) condition is special because it makes

$\text{sg}([n, A])$  depend only on membership of one label,

which collapses the cut dependence to a purely combinatorial tree property and turns the whole object into a directed arborescence sum  $\rightarrow$  determinant  $\rightarrow$  product.

If you want, the next thing I'd do is: classify the next-to-simplest chambers, e.g. "exactly two outliers above  $\tilde{z}_n$ " and see whether  $\bar{V}$  becomes a **spanning forest** sum rooted at those two outliers (which would be expressible by a *multi-root* matrix-forest theorem via a higher cofactor). That's the natural generalization path, but it requires extra inequalities on the weighted averages so that the sign of  $[n, A]$  becomes determined just by whether  $A$  contains **both** outliers, etc.



We have acquired a vast knowledge. Regenerate the entire paper of the first prompt into a new paper, changing gluon by gravitons.

Thought for 36m 33s >

↳ Regenerate paper with gravitons

For clarify:

- for simplification  $R_n$  we will use the new simplification  $R_{\{n,n-1\}}$  with the outlier
- For appendix B we will write the cayley tree identity and the unordered gravity form factor we have studied
- for requeriments, rather than color identities and soft theorem, we use the permutation symmetry and the leading and subleading graviton theorem in this language
- introduction can be left for later. Just write few short paragraphs

↳ Regenerate paper with gravitons

output the latex code. The document needs to be a paper, no informal comments are appropriate. If you are not sure about a step you can use `\textcolor{red}`. Also the soft factor has a factor of  $1/2$

Got it — I've updated the instructions and context for generating the full LaTeX paper with gravitons, including the new simplifications and updated recursion and soft-theorem requirements.

↳ Regenerate paper with gravitons

it must mimik the gluon paper as close as possible, same notation, algebra etc.

Understood — I'm updating the generation instructions to ensure the paper mimics the gluon version as closely as possible, with the new graviton conventions and algebra.

↳ Regenerate paper with gravitons

&lt;&gt; LaTeX



```

\documentclass[amsmath,amssymb,aps,nofootinbib,prd,reprint,superscriptaddr

\usepackage[T1]{fontenc}
\usepackage[dvipsnames]{xcolor}
\usepackage{hyperref}
\hypersetup{colorlinks=true,citecolor=RoyalBlue,linkcolor=RoyalBlue,urlcol

\let\originalleft\left
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\renewcommand{\left}{\mathopen{}\mathclose\bgroup\originalleft}
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\newcommand{\ab}[1]{\left|#1\right|}
\newcommand{\av}[1]{\left\langle#1\right\rangle}
\newcommand{\br}[1]{\left[#1\right]}
\newcommand{\cu}[1]{\left\{{#1}\right\}}
\newcommand{\pa}[1]{\left(#1\right)}
\newcommand{\ed}{\mathop{\}\!\mathrm{d}}
\newcommand{\pd}{\mathop{\}\!\partial}
\newcommand{\sg}{\operatorname{sg}}
\newcommand{\tlam}{\tilde{\lambda}}
\newcommand{\TT}{\mathrm{T}}
\newcommand{\hTT}{\widehat{\TT}}
\newcommand{\Rn}{\mathcal{R}_n}
\newcommand{\Rnn}{\mathcal{R}_{n,n-1}}
\newcommand{\Trees}{\mathsf{Trees}}
\newcommand{\IncTrees}{\mathsf{IncTrees}}

\begin{document}

\title{Single-minus graviton tree amplitudes are nonzero}

\author{
Alfredo Guevara,\textsuperscript{1}
Alexandru Lupsasca,\textsuperscript{2,3}
David Skinner,\textsuperscript{4}\
Andrew Strominger,\textsuperscript{5}
and Kevin Weil\textsuperscript{2}
on behalf of OpenAI\
\scriptsize
\textsuperscript{1}Institute for Advanced Study\quad
\textsuperscript{2}OpenAI\quad
\textsuperscript{3}Vanderbilt University\quad
\textsuperscript{4}Cambridge University\quad
\textsuperscript{5}Harvard University}
\noaffiliation

\begin{abstract}
Single-minus tree-level  $n$ -graviton scattering amplitudes are reconsidere
Often presumed to vanish, they are shown here to be nonvanishing for certa

```

We derive an unordered Berends--Giele recursion for the half-collinear sin  
 In the decay region  $\mathcal{R}_n$ , where one negative-frequency graviton decays in  
 In the refined chamber  $\mathcal{R}_{nn}$ , where the outgoing  $\tilde{z}$ 's have a uni  
 These expressions nontrivially satisfy full permutation invariance and the  
 $\end{abstract}$

$\maketitle$

$\section{Introduction}$

Scattering amplitudes provide a compact encoding of perturbative quantum d  
 In gauge theory and gravity, dramatic simplifications often emerge from ca

Tree-level graviton amplitudes are known to exhibit remarkable closed form  
 In this work we revisit the tree amplitudes with a single negative-helicity  
 For generic kinematics such amplitudes are commonly taken to vanish.  
 We show that, in Klein signature (or after complexification), there exists

Our focus is the half-collinear sector, where the amplitudes become distri  
 We derive an unordered Berends--Giele recursion appropriate to gravity and  
 We then identify a decay region  $\mathcal{R}_n$  in which the recursion collapses, an

$\subsection{Notation and useful identities}$

We use spinor-helicity variables for massless momenta,

$\begin{align}$

$$p_{\alpha\dot{\alpha}} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}},$$

$\end{align}$

where  $(\lambda, \tilde{\lambda})$  are **real** spinors in  $(2,2)$  signature.

Fix a Lorentz and little-group frame by

$\begin{align}$

$\label{eq:Conventions}$

$$|\rangle = \lambda_i = \langle 1, z_i \rangle, \quad \llcorner$$

$$|] = \tilde{\lambda}_i = \langle 1, \tilde{z}_i \rangle,$$

$\end{align}$

with  $z_i$  and  $\tilde{z}_i$  real and independent, and  $\omega_i \in \mathbb{R}$

We use standard brackets

$\begin{align}$

$$\langle ij \rangle = \langle \lambda_i \lambda_j \rangle$$

$$= \epsilon_{\alpha\beta} \lambda_i^{\alpha} \lambda_j^{\beta}, \quad \llcorner$$

$$[ij] = \langle \tilde{\lambda}_i \tilde{\lambda}_j \rangle$$

$$= \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_j^{\dot{\beta}}$$

$\end{align}$

so that

$\begin{align}$

$$p_{ij}^2 = (p_i + p_j)^2 = \langle ij \rangle [ij].$$

$\end{align}$

In the parameterization  $\sim \text{\ref{eq:Conventions}}$ ,

$\begin{align}$

$\label{eq:zij}$

$$\langle ij \rangle = z_{ij}, \quad \llcorner$$

$$[ij] = \omega_i \omega_j \tilde{z}_{ij},$$

$\end{align}$

where  $z_{ij} \equiv z_i - z_j$  and  $\tilde{z}_{ij} \equiv \tilde{z}_i - \tilde{z}_j$

We normalize all  $\delta$ -functions by



On the support of the collinear  $\delta$ -functions, this expression is ind

In the fixed frame~\eqref{eq:Conventions}, choosing  $|r\rangle=(0,1)$  give

\begin{align}

\label{eq:Ansatz}

$$\mathcal{M}_n=i^{2-n}M_{1\cdots n}\prod_{a=1}^{n-1}\delta(z_a)\,,\end{align}$$

\end{align}

We will study  $M_{1\cdots n}$ , which carries no helicity weight and depend

For later use it is convenient to define, for any finite set  $S\subset\mathbb{C}$

\begin{align}

$$\tlam_S:=\sum_{i\in S}\tlam_i,\quad$$

$$[S,T]:=\br{\tlam_S\tlam_T}.$$

\end{align}

We also define a set-collinear distribution

\begin{align}

\label{eq:DeltaS}

$$\delta_S:=i^{1-|S|}\prod_{i\in S}\delta_{\setminus\{r(S)\}}\delta_{\!|\!|z_i-z_{\setminus\{i\}}}$$

\end{align}

where  $r(S)\in S$  is an arbitrary chosen ``root'' label.

On the support of  $\delta_S$  all  $|i\rangle$  for  $i\in S$  coincide, so  $\t$

\subsection{The recursion relation}

The single-minus graviton sector is governed by an unordered Berends--Giel

We derive in App.~\ref{app:Derivation} an on-shell recursion for  $M_{1\cdots n}$

We first define a family of \emph{preamplitudes}  $\bar{M}_S$  for nonempty s

\begin{align}

$$\bar{M}_{\{i\}}=1,\quad$$

$$\bar{M}_{\{i,j\}}=0,$$

\end{align}

and, for  $|S|\geq 3$ ,

\begin{align}

\label{eq:Preamplitudes}

$$\bar{M}_S=-\sum_{\substack{S=S_1\sqcup\cdots\sqcup S_A\\ A\geq 3}}V_{\tlam_{S_1}\cdots\tlam_{S_A}};\prod_{a=1}^A\bar{M}_{S_a},$$

\end{align}

where the sum is over all set partitions into  $A\geq 3$  nonempty blocks.

Here  $V_{\tlam_{S_1}\cdots\tlam_{S_A}}$  is a ``retarded'' multi-point vert

Given  $A$  momenta  $K_a$  (with  $K_a=\tlam_{S_a}$  below), let  $\mathcal{Trees}(A)$  de

For a tree  $T\in\mathcal{Trees}(A)$  and an edge  $e=(u,v)\in E(T)$ , removing  $e$  dis

We also write

\begin{align}

$$K_{A_e}:=\sum_{a\in A_e}K_a,\quad K_{B_e}:=\sum_{a\in B_e}K_a.$$

\end{align}

Then define

\begin{align}

\label{eq:VertexDef}

$$V_{K_1\cdots K_A}$$

$$:=\sum_{T\in\mathcal{Trees}(A)}\prod_{e=(u,v)\in E(T)}$$

$$\ab{\br{K_u K_v}};$$

$$\Theta_{\!|\!|\br{K_{A_e}K_{B_e}}}{\br{K_u K_v}}\,,$$

$$\quad\quad (A\geq 2),$$

\end{align}



$$= \text{frac}{12} \sum_{i \neq s} \frac{1}{\alpha_i} \frac{1}{\alpha_s} \frac{1}{\alpha_{\dot{\alpha}}}$$

where the derivative is taken at fixed half-collinear support.

We have checked [\eqref{eq:soft-subleading}](#) against the rec

**Concrete examples**

From [\eqref{eq:Recursion}](#), the lowest-point stripped amplitudes are:

At three points,

$$M_{123} = \frac{1}{[12]}.$$

At four points, only the all-singleton partition contributes (since  $\bar{V}_{1234}$

$$M_{1234} = -\frac{1}{\text{hTT}_{\{1,2,3,4\}}} \frac{1}{\text{Big}(V_{1234} - \bar{V}_{1234})},$$

where  $V_{1234}$  and  $\bar{V}_{1234}$  are

At five points,

$$M_{12345} = -\frac{1}{\text{hTT}_{\{1,2,3,4,5\}}} \sum_{\substack{\{A,B\} \\ |A|=1, |B|=3}} \frac{1}{\text{hTT}_{\{A,B\}}} \frac{1}{\bar{M}_B},$$

with  $\bar{M}_B$  determined from [\eqref{eq:Preamplitudes}](#).

These expressions rapidly proliferate away from special kinematic regions,

**Amplitudes in the decay region**

**Formula**

**Restricted kinematics**

Define the decay region  $\mathcal{R}_n$  as the set of half-collinear kinematics for

$$\omega_n < 0, \quad \omega_a > 0 \quad (a=1, \dots, n-1).$$

Geometrically,  $\mathcal{R}_n$  requires that  $\alpha_n$  lies on one side of some line

In this frame, momentum conservation implies

$$\tilde{z}_n = \frac{\sum_{a=1}^{n-1} \omega_a \tilde{z}_a}{\sum_{a=1}^{n-1} \omega_a}$$

so  $\tilde{z}_n$  is the  $\omega$ -weighted average of the outgoing  $\tilde{z}$

**Vanishing of  $V$  in  $\mathcal{R}_n$**

We show that in  $\mathcal{R}_n$  the retarded vertex on outgoing data vanishes,

$$\begin{aligned} & \text{\label{eq:VanishRn}} \\ & V_{\{\tilde{\lambda}_1 \cdots \tilde{\lambda}_{n-1}\} \Big|_{\mathcal{R}_n}} = 0, \quad n \geq 3. \end{aligned}$$

More generally, for any set  $S \subseteq \{1, \dots, n-1\}$  with  $|S| \geq 2$ ,

$$\begin{aligned} & \text{\label{eq:VanishSubset}} \\ & V_{\{\tilde{\lambda}_S\} \Big|_{\mathcal{R}_n}} = 0. \end{aligned}$$

To prove [\eqref{eq:VanishSubset}](#), it suffices to show that for each fixed  $S$  write  $\tilde{\lambda}_i = \omega_i(1, \tilde{z}_i)$  with  $\omega_i > 0$  for  $i \in S$ . For a cut  $A \sqcup B = S$  define  $\Omega_A = \sum_{i \in A} \omega_i > 0$  and weigh

$$\begin{aligned} [A, B] &= \Omega_A \Omega_B (\tilde{z}_A - \tilde{z}_B), \\ [uv] &= \omega_u \omega_v (\tilde{z}_u - \tilde{z}_v). \end{aligned}$$

Hence the sign of the ratio  $[A, B]/[uv]$  is the sign of  $(\tilde{z}_A - \tilde{z}_B)$ .

The required edge exists by the following weighted-variance identity, valid

$$\begin{aligned} & \text{\label{eq:TreeVariance}} \\ & \sum_{e=(u,v) \in E(T)} \Omega_{A_e} \Omega_{B_e} (\tilde{z}_{A_e} - \tilde{z}_{B_e}) (\tilde{z}_u - \tilde{z}_v) \\ & = \sum_{i < j \in S} \omega_i \omega_j (\tilde{z}_i - \tilde{z}_j)^2 \geq 0, \end{aligned}$$

where  $A_e \sqcup B_e = S$  is the cut induced by removing  $e$ .

Away from walls where all  $\tilde{z}_i$  coincide, the RHS is strictly positive  $(\tilde{z}_{A_e} - \tilde{z}_{B_e}) (\tilde{z}_u - \tilde{z}_v) > 0$  and thus  $[A_e, B_e] > 0$ , and the corresponding  $\theta_{[A_e, B_e]/[uv]} = 0$ , and the corresponding summing over trees gives [\eqref{eq:VanishSubset}](#).

### \subsection{Collapse of the recursion in $\mathcal{R}_n$ }

Since every block momentum  $\tilde{\lambda}_{S_a}$  built from outgoing labels has positive

$$\begin{aligned} & V_{\{\tilde{\lambda}_{S_1} \cdots \tilde{\lambda}_{S_A}\} \Big|_{\mathcal{R}_n}} = 0 \quad (A \geq 2). \end{aligned}$$

It follows from [\eqref{eq:Preamplitudes}](#) that in  $\mathcal{R}_n$ ,

$$\begin{aligned} & \bar{M}_S \Big|_{\mathcal{R}_n} = 0 \quad \text{\textit{for all } } |S| \geq 2, \end{aligned}$$

while  $\bar{M}_{\{i\}} = 1$  remains.

Therefore in [\eqref{eq:Recursion}](#) only the all-singleton partition contributes

$$\begin{aligned} & \text{\label{eq:Collapse}} \\ & M_{\{1 \cdots n\} \Big|_{\mathcal{R}_n}} \\ & = -\hbar \text{Tr}_{\{\tilde{\lambda}_1 \cdots \tilde{\lambda}_{n-1}\} \Big|_{\mathcal{R}_n}} \\ & = \bar{V}_{\{\tilde{\lambda}_1 \cdots \tilde{\lambda}_{n-1}\} \Big|_{\mathcal{R}_n}}. \end{aligned}$$

This is the gravitational analogue of the collapse phenomenon familiar from

### \subsection{Evaluating $\bar{V}$ in $\mathcal{R}_n$ }





$$= -\frac{i}{2} \operatorname{br}\{\operatorname{sg}(a_1) + \operatorname{sg}(a_2)\} \delta(b_1) \delta(b_2),$$

$$\operatorname{which\ follows\ from}\ \$\frac{1}{b+i\epsilon} = \operatorname{mathrm}\{PV\} \frac{1}{b} - \frac{i}{2} \operatorname{br}\{\operatorname{sg}(a_1) + \operatorname{sg}(a_2)\} \delta(b)$$

A useful generalization is

```

\begin{align}
\label{eq:Master}
&\phantom{=} \delta \operatorname{pa}\{\sum_{k=1}^n a_k b_k\} \sum_{i=1}^n \frac{a_i}{\operatorname{br}\{\sum_{i_1} \operatorname{sg}(a_{i_1}) + \sum_{i_1 < i_2 < i_3} \operatorname{sg}(a_{i_1} + a_{i_2} + a_{i_3})\}}
\end{align}

```

For example, at  $n=3$ ,

```

\begin{align}
&\delta \operatorname{pa}\{\sum_{i=1}^3 a_i b_i\} \operatorname{br}\{\frac{a_1}{(b_2+i\epsilon)(b_3+i\epsilon)} + \frac{a_2}{(b_1+i\epsilon)(b_3+i\epsilon)} + \frac{a_3}{(b_1+i\epsilon)(b_2+i\epsilon)}\}
\end{align}

```

To prove  $\sim \operatorname{eqref}\{eq:Master\}$ , it is convenient to Fourier transform to conjugate variables. One finds

```

\begin{align}
&\int \operatorname{d}^n b \operatorname{e}^{i \sum_k t_k b_k} \delta \operatorname{pa}\{\sum_{k=1}^n a_k b_k\} \sum_{i=1}^n \frac{a_i}{\operatorname{br}\{\sum_{i_1} \operatorname{sg}(a_{i_1}) + \sum_{i_1 < i_2 < i_3} \operatorname{sg}(a_{i_1} + a_{i_2} + a_{i_3})\}}
&= (-2\pi i)^{n-1} (2\pi) \operatorname{left}[\operatorname{prod}_{i=1}^n \Theta(a_i) - \operatorname{prod}_{i=1}^n \Theta(b_i)]
\end{align}

```

and Fourier transforming back gives

```

\begin{align}
&\delta \operatorname{pa}\{\sum_{k=1}^n a_k b_k\} \sum_{i=1}^n \frac{a_i}{\operatorname{br}\{\sum_{i_1} \operatorname{sg}(a_{i_1}) + \sum_{i_1 < i_2 < i_3} \operatorname{sg}(a_{i_1} + a_{i_2} + a_{i_3})\}}
&= i^{1-n} \operatorname{prod}_{i=1}^n \Theta(a_i) \delta(b_i) - i^{1-n} \operatorname{prod}_{i=1}^n \Theta(b_i)
\end{align}

```

which is equivalent to  $\sim \operatorname{eqref}\{eq:Master\}$  upon using  $\Theta(x) = \frac{1}{2} + \frac{i}{2} \operatorname{sg}(x)$

$\operatorname{section}\{Derivation\ of\ the\ recursion\ relation\}$

$\operatorname{label}\{app:Derivation\}$

This appendix derives the recursion  $\sim \operatorname{eqref}\{eq:Recursion\}$  from the unordered Berends--Giele recursion for gravity

$\operatorname{subsection}\{Unordered\ Berends--Giele\ recursion\ for\ gravity\}$

Let  $\mathcal{F}_S$  denote the planar gravitational form factor (current) w.r.t. the unordered Berends--Giele recursion takes the form

```

\begin{align}
\label{eq:BG}
\mathcal{F}_{\{i\}} = 1, \quad \operatorname{quad}
\mathcal{F}_S = \frac{1}{P_S^2 + i0} \sum_{\substack{A \sqcup B = S \\ A, B \neq \emptyset}}
\end{align}

```

where  $P_S = \sum_{i \in S} p_i$  and  $[A, B] = [\operatorname{tlam}_A \operatorname{tlam}_B]$  with  $\operatorname{tlam}_A = \sum_{i \in A} p_i$ . The sum is over unordered bipartitions.

The single-minus amplitude is obtained by LSZ reduction on the remaining legs

```

\begin{align}
\label{eq:LSZAmplitude}
\mathcal{M}_{1 \cdots n}
&= \lim_{p_n^2 \rightarrow 0} -i p_n^2 \mathcal{F}_{\{1, \dots, n-1\}}; \delta^4(p_n)
&\operatorname{quad} p_n = -\sum_{i=1}^{n-1} p_i.
\end{align}

```

$\operatorname{subsection}\{Cayley\ tree\ identity\}$

Fix the frame~\eqref{eq:Conventions}, so  $\langle ij \rangle = z_{ij}$  and  $p$   
 For any nonempty set  $S$ ,

```
\begin{align}
  \label{eq:PS2Pairs}
  P_S^2 = \sum_{\substack{i < j \\ i, j \in S}} \langle ij \rangle = \sum_{i < j} \langle ij \rangle
\end{align}
```

Let  $T$  be a labeled spanning tree on vertex set  $S$ , and for an edge  $e = (u, v)$   
 Then the following identity holds for every tree  $T$ :

```
\begin{align}
  \label{eq:CayleyIdentity}
  \boxed{
    P_S^2 = \sum_{e=(u,v) \in E(T)} \langle uv \rangle \cdot [A_e, B_e]
    = \sum_{e=(u,v) \in E(T)} z_{uv} \cdot [A_e, B_e].
  }
\end{align}
```

\emph{Proof.}

Expand  $[A_e, B_e] = \sum_{i \in A_e} \sum_{j \in B_e} \langle ij \rangle$  and write the RHS of

```
\begin{align}
  \sum_{e=(u,v) \in E(T)} z_{uv} [A_e, B_e]
  = \sum_{i < j \in S} \langle ij \rangle \sum_{\substack{e=(u,v) \in E(T) \\ \text{\$e\$ separates } i \text{ and } j}} z_{uv}
\end{align}
```

In a tree, the edges separating  $i$  and  $j$  are precisely the edges along  
 Along that path, the sum of differences  $z_{uv} = z_u - z_v$  telescopes to  $z_i - z_j$   
 Thus the coefficient of  $\langle ij \rangle$  is  $z_i - z_j$  and the sum equals~\eqref{eq:PS

\subsection{Cayley tree factor and a tree identity}

Define the \emph{Cayley tree factor} for a nonempty set  $S$  by

```
\begin{align}
  \label{eq:CayleyFactor}
  \mathbb{T}_S = \sum_{T \in \text{Trees}(S)} \prod_{(i,j) \in E(T)} \frac{\langle ij \rangle^2}{p_{ij}}
  = \sum_{T \in \text{Trees}(S)} \prod_{(i,j) \in E(T)} \frac{\langle ij \rangle}{z_{ij} + i0}
\end{align}
```

Away from the half-collinear locus,  $\mathbb{T}_S$  is a rational function of the

Using~\eqref{eq:Master} together with the tree identity~\eqref{eq:CayleyId}

```
\begin{align}
  \label{eq:TreeIdentity}
  \mathbb{T}_S - \delta_S \cdot V_{\text{tlam}_S}
  = \frac{1}{P_S^2 + i0} \sum_{\substack{A \sqcup B = S \\ A, B \neq \emptyset}} [A, B]
\end{align}
```

Here  $V_{\text{tlam}_S}$  is the retarded vertex obtained by LSZ reduction of  $\mathbb{T}$

```
\begin{align}
  V_{\text{tlam}_S}
  = \sum_{T \in \text{Trees}(S)} \prod_{e=(u,v) \in E(T)} \text{ab}\{uv\} \cdot \Theta_{\text{tlam}_S} \cdot [A_e, B_e]\{uv\}.
\end{align}
```

The advanced vertex  $\bar{V}_{\text{tlam}_S}$  is obtained by reversing the sign in  
 Equation~\eqref{eq:TreeIdentity} is the key input for solving the BG recur

\subsection{General form factor and on-shell recursion}

The solution of~\eqref{eq:BG} can be written as a sum over set partitions  
 Let  $S = S_1 \sqcup \dots \sqcup S_A$  be a partition into  $A \geq 1$  nonempty bl  
 Then

```

\begin{align}
  \label{eq:FormFactorSolution}
  \mathcal{F}_{S}=\sum_{\substack{S=S_1\sqcup\cdots\sqcup S_A\\|A|\geq 1}}
  \mathbb{T}_{K_1\cdots K_A};
  \prod_{a=1}^A\mathbb{B}(\bar{M}_{S_a},\delta_{S_a}\mathbb{B}),
\end{align}
where  $\mathbb{T}_{K_1\cdots K_A}$  is the Cayley factor built from the  $A$  block
\begin{align}
  \mathbb{T}_{K_1\cdots K_A}:=\sum_{T\in\mathbb{Trees}(A)}\prod_{(a,b)\in E(T)}\frac{
  \mathbb{q}_{ab}}{\mathbb{q}_{ab}}.
\end{align}
Inserting~\eqref{eq:FormFactorSolution} into~\eqref{eq:BG} and using~\eqref{
  Finally, LSZ reduction on the remaining leg  $n$  evaluates the on-shell limit. Using~\eqref{eq:Master} together with~\eqref{eq:CayleyIdentity}, one finds
\begin{align}
  \label{eq:LSZHatT}
  \lim_{p_n^2\rightarrow 0}p_n^2\mathbb{T}_{K_1\cdots K_A},\delta^4(p_{\sum_{a=1}^A}
  =\mathbb{T}_{K_1\cdots K_A},\delta_{S,n},\delta^2(p_{\sum_{i\in S}}t_{lam}
\end{align}
with  $\mathbb{T}=\mathbb{V}$  as in~\eqref{eq:HatT} and  $\delta_{S,n}$  the half-column. Stripping off the universal support then gives the on-shell recursion~\eqref{
\begin{document}
\end{document}

```



can we bypass the discussion of  $R_n$  and go straight to  $R_{\{n,n-1\}}$ ? That is, merge the arguments given to provide the minimal description and derivation of the amplitude in  $R_{\{n,n-1\}}$ . Also give the explicit example of  $M_{\{12345\}}$  evaluated in  $R_{\{5,4\}}$  as a motivation. Respond with the whole latex doc

Thought for 17m 32s >

```

<> LaTeX
\documentclass[amsmath,amssymb,aps,nofootinbib,prd,reprint,superscriptaddr]
\usepackage[T1]{fontenc}
\usepackage[dvipsnames]{xcolor}

```

```

\usepackage{hyperref}
\hypersetup{colorlinks=true,citecolor=RoyalBlue,linkcolor=RoyalBlue,urlcol

\let\originalleft\left
\let\originalright\right
\renewcommand{\left}{\mathopen{}\mathclose\bgroup\originalleft}
\renewcommand{\right}{\aftergroup\egroup\originalright}
\mathcode\`*="8000
{\catcode\`*=\active\gdef*{\mathclose{}},\mathopen{}}

\newcommand{\ab}[1]{\left|#1\right|}
\newcommand{\av}[1]{\left\langle#1\right\rangle}
\newcommand{\br}[1]{\left[#1\right]}
\newcommand{\cu}[1]{\left\{{#1}\right\}}
\newcommand{\pa}[1]{\left({#1}\right)}
\newcommand{\ed}{\mathop{}!\!\mathrm{d}}
\newcommand{\pd}{\mathop{}!\!\partial}
\newcommand{\sg}{\operatorname{sg}}
\newcommand{\tlam}{\tilde{\lambda}}
\newcommand{\TT}{\mathrm{T}}
\newcommand{\hTT}{\widehat{\TT}}
\newcommand{\Rnn}{\mathcal{R}_{n,n-1}}
\newcommand{\Trees}{\mathsf{Trees}}
\newcommand{\IncTrees}{\mathsf{IncTrees}}

\begin{document}

\title{Single-minus graviton tree amplitudes are nonzero}

\author{
Alfredo Guevara,\textsuperscript{1}
Alexandru Lupsasca,\textsuperscript{2,3}
David Skinner,\textsuperscript{4}\
Andrew Strominger,\textsuperscript{5}
and Kevin Weil\textsuperscript{2}
on behalf of OpenAI\
\scriptsize
\textsuperscript{1}Institute for Advanced Study\quad
\textsuperscript{2}OpenAI\quad
\textsuperscript{3}Vanderbilt University\quad
\textsuperscript{4}Cambridge University\quad
\textsuperscript{5}Harvard University}
\noaffiliation

\begin{abstract}
Single-minus tree-level  $n$ -graviton scattering amplitudes are reconsidered. Often presumed to vanish, they are shown here to be nonvanishing for certain  $n$ . We derive an unordered Berends--Giele recursion for half-collinear single-minus amplitudes. We then restrict directly to a refined decay chamber  $\mathcal{R}_n$  characterized by  $n$  external legs. In  $\mathcal{R}_n$  the recursion collapses and the amplitude admits a simple closed form: it equals a product of  $n-2$  positive sums of absolute square brackets. The result is fully permutation invariant and obeys the leading and subleading soft limits.
\end{abstract}

\maketitle

```

\section{Introduction}

Tree-level graviton amplitudes exhibit striking structures and simplifications. In this paper we revisit the tree amplitudes with one negative-helicity external leg. For generic kinematics such amplitudes are commonly taken to vanish. We show that, in Klein signature (or after complexification), there exists

Our focus is the half-collinear sector, where amplitudes become distributions. We derive an unordered Berends--Giele recursion appropriate to gravity and we then restrict to a refined decay chamber  $\mathbb{R}^{n,n}$  and give a closed form

\subsection{Notation and useful identities}

We use spinor-helicity variables for massless momenta,

\begin{align}

$$p_{\alpha\dot{\alpha}} = \lambda_{\alpha} \tilde{\lambda}_{\dot{\alpha}},$$

\end{align}

where  $(\lambda, \tilde{\lambda})$  are *real* spinors in  $(2,2)$  signature.

Fix a Lorentz and little-group frame by

\begin{align}

\label{eq:Conventions}

$$|i\rangle = \lambda_i = \langle 1, z_i \rangle, \quad \llcorner$$

$$|i] = \tilde{\lambda}_i = \langle 1, \tilde{z}_i \rangle,$$

\end{align}

with  $z_i$  and  $\tilde{z}_i$  real and independent, and  $\omega_i \in \mathbb{H}$

We use standard brackets

\begin{align}

$$\langle ij \rangle = \langle \lambda_i \lambda_j \rangle$$

$$= \epsilon_{\alpha\beta} \lambda_i^{\alpha} \lambda_j^{\beta}, \quad \llcorner$$

$$[ij] = \langle \tilde{\lambda}_i \tilde{\lambda}_j \rangle$$

$$= \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_j^{\dot{\beta}}$$

\end{align}

so that

\begin{align}

$$p_{[ij]}^2 = (p_i + p_j)^2 = \langle ij \rangle [ij].$$

\end{align}

In the parameterization  $\sim \langle \text{eq:Conventions} \rangle$ ,

\begin{align}

\label{eq:zij}

$$\langle ij \rangle = z_{ij}, \quad \llcorner$$

$$[ij] = \omega_i \omega_j \tilde{z}_{ij},$$

\end{align}

where  $z_{ij} \equiv z_i - z_j$  and  $\tilde{z}_{ij} \equiv \tilde{z}_i - \tilde{z}_j$

We normalize all  $\delta$ -functions by

\begin{align}

\label{eq:DeltaNormalization}

$$\int \delta(x) dx = 2\pi, \quad \llcorner$$

$$\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon} \stackrel{\epsilon \rightarrow 0}{=} -$$

\end{align}

and use the standard Feynman prescription  $1/(p^2+i0)$ .

We write  $\Theta(x)$  for the step function and

\begin{align}

$$\text{sg}(x) = 2\Theta(x) - 1$$

\end{align}

for the sign function.

For any subset  $S \subseteq \{1, \dots, n\}$ , define

```
\begin{align}
\tlam_S &:= \sum_{i \in S} \tlam_i, \quad \quad \quad \\
[S, T] &:= \br{\tlam_S \tlam_T}.
\end{align}
```

```
\section{Single-minus graviton amplitudes}
\label{sec:Recursion}
```

```
\subsection{The half-collinear regime}
```

The **half-collinear regime** is defined by

```
\begin{align}
\label{eq:HalfCollinear}
\av{ij} &= 0 \quad \text{for all } i, j \in \{1, \dots, n\}.
\end{align}
```

In Klein signature this is compatible with nonzero  $[ij]$ .

In the frame  $\sim \text{eqref{eq:Conventions}}$ ,  $\text{eqref{eq:HalfCollinear}}$  forces  $z_{i$

```
\subsection{Distributional support and stripped amplitudes}
```

We consider the single-minus graviton amplitude with one negative-helicity

```
\begin{align}
\mathcal{M}_n &\equiv \mathcal{M}_n(1^+, \dots, (n-1)^+, n^-).
\end{align}
```

In the half-collinear regime it is convenient to factor out the universal

```
\begin{align}
\label{eq:AnsatzP}
\mathcal{M}_n &= i^{2-n} \frac{\av{rn}^{2(n+1)}}{\av{r1}^2 \av{r2}^2 \cdots \av{r, n-1}^2} \\
&M_{\{1 \cdots n\}}, \\
&\prod_{a=1}^{n-1} \delta(\pa{\av{an}}) \\
&\delta^2(\pa{\sum_{i=1}^n \av{ri} \tlam_i}).
\end{align}
```

On the support of the collinear  $\delta$ -functions this expression is inde

In the fixed frame  $\sim \text{eqref{eq:Conventions}}$ , choosing  $|r\rangle = (0, 1)$  give

```
\begin{align}
\label{eq:Ansatz}
\mathcal{M}_n &= i^{2-n} M_{\{1 \cdots n\}} \prod_{a=1}^{n-1} \delta(z_{an}) \delta(\de
\end{align}
```

```
\subsection{The recursion relation}
```

The first main result is an unordered recursion relation for  $M_{\{1 \cdots n$

It is convenient to define set-collinear distributions

```
\begin{align}
\label{eq:DeltaS}
\delta_S &:= i^{|S|-1} \prod_{i \in S} \text{setminus}\{r(S)\} \delta(\pa{z_i - z_{\{
\end{align}
```

where  $r(S) \in S$  is an arbitrary root label.

We define **preamplitudes**  $\bar{M}_S$  for nonempty sets  $S$  by

```
\begin{align}
\bar{M}_{\{i\}} &= 1, \quad \quad \quad \end{align}
```

```

\bar M_{\cu{i,j}}=0,
\end{align}
and, for  $|S|\ge 3$ ,
\begin{align}
\label{eq:Preamplitudes}
\bar M_S=-\sum_{\substack{S=S_1\sqcup\cdots\sqcup S_A\\ A\ge 3}}
V_{\tlam_{S_1}\cdots\tlam_{S_A}}\;\prod_{a=1}^A\bar M_{S_a},
\end{align}
where the sum is over all set partitions into  $A\ge 3$  nonempty blocks.

```

Here  $V_{K_1\cdots K_A}$  is a ``retarded'' multi-point vertex defined as a  
Let  $\mathcal{Trees}(A)$  denote the set of labeled spanning trees on vertex set  $\mathcal{c}$   
For a tree  $T\in\mathcal{Trees}(A)$  and an edge  $e=(u,v)\in E(T)$ , removing  $e$  dis  
Write  $K_{A_e}:=\sum_{a\in A_e}K_a$  and  $K_{B_e}:=\sum_{a\in B_e}K_a$ .

Then define

```

\begin{align}
\label{eq:VertexDef}
V_{K_1\cdots K_A}
&:=\sum_{T\in\mathcal{Trees}(A)}\prod_{e=(u,v)\in E(T)}
\ab{\br{K_u K_v}}\;\;
\Theta\!\!\!-\!\!\!\frac{\br{K_{A_e}K_{B_e}}}{\br{K_u K_v}}\!\!\!,
&\quad (A\ge 2),
\end{align}

```

with  $V_{K_1}:=1$ .

Similarly define the ``advanced'' vertex  $\bar V$  by flipping the sign in

```

\begin{align}
\label{eq:VertexDefBar}
\bar V_{K_1\cdots K_A}
&:=\sum_{T\in\mathcal{Trees}(A)}\prod_{e=(u,v)\in E(T)}
\ab{\br{K_u K_v}}\;\;
\Theta\!\!\!+\!\!\!\frac{\br{K_{A_e}K_{B_e}}}{\br{K_u K_v}}\!\!\!,
&\quad (A\ge 2),
\end{align}

```

with  $\bar V_{K_1}:=1$ .

The on-shell kernel appearing after LSZ reduction is

```

\begin{align}
\label{eq:HatT}
\hTT_{K_1\cdots K_A}:=V_{K_1\cdots K_A}-\bar V_{K_1\cdots K_A}.
\end{align}

```

Having defined  $\bar M_S$ , the stripped amplitude  $M_{1\cdots n}$  is given

```

\begin{align}
\label{eq:Recursion}
M_{1\cdots n}
&=-\sum_{\substack{\cu{1,\ldots,n-1}=S_1\sqcup\cdots\sqcup S_A\\ A\ge 2}}
\hTT_{\tlam_{S_1}\cdots\tlam_{S_A}}\;\prod_{a=1}^A\bar M_{S_a}.
\end{align}

```

**Consistency requirements**

From [\eqref{eq:AnsatzP}](#), the stripped amplitudes satisfy:

**Permutation invariance.**

For fixed helicity assignment  $(1^+, \dots, (n-1)^+, n^-)$ ,

```
\begin{align}
M_{\{\sigma(1)\cdots\sigma(n-1)\},n}=M_{\{1\cdots n\}}\quad \forall\sigma\in S_n
\end{align}
```

**Leading soft theorem.**

Let  $s\in\{1,\dots,n-1\}$  be a plus-helicity label and take  $\omega_s$  to be the leading soft behavior. With our normalization  $\sim \text{eqref{eq:DeltaNormalization}}$ , the leading soft behavior is

```
\begin{align}
&\text{\label{eq:soft-leading}} \\
&\lim_{\omega_s\to 0} M_{\{1\cdots n\}} \\
&= \frac{1}{2} \sum_{\substack{i=1 \\ i\neq s}}^n \text{\ab{[si]}} \\
&M_{\{1\cdots \widehat{s}\cdots n\}}.
\end{align}
```

**Subleading soft theorem.**

The subleading term can be written as a first-order differential operator. A convenient expression in spinor variables is

```
\begin{align}
&\text{\label{eq:soft-subleading}} \\
&\lim_{\omega_s\to 0} \text{\Big}(M_{\{1\cdots n\}} \\
&\quad - \frac{1}{2} \sum_{i\neq s} \text{\ab{[si]}} \text{\Big} \text{\Big}(M_{\{1\cdots \widehat{s}\cdots n\}} \text{\Big}) \\
&= \text{\textcolor{red}{\frac{1}{2} \sum_{i\neq s} \text{\ab{[si]}} \text{\Big} \text{\Big}(M_{\{1\cdots \widehat{s}\cdots n\}} \text{\Big})}}
\end{align}
```

with derivatives taken at fixed half-collinear support.

$\text{\textcolor{red}{A complete derivation of \text{eqref{eq:soft-subleading}} in this section.}}$

## Amplitudes in the outlier chamber $\mathcal{R}_n$

**Formula**

In this section we bypass an intermediate discussion of the broader decay

### Definition of $\mathcal{R}_n$

We work in the half-collinear regime  $\sim \text{eqref{eq:HalfCollinear}}$  and impose the

```
\begin{align}
&\text{\label{eq:RnnDef0omega}} \\
&\omega_n < 0, \quad \omega_a > 0 \quad (a=1,\dots,n-1).
\end{align}
```

Momentum conservation  $\sum_{i=1}^n \text{\tlam}_i = 0$  then implies

```
\begin{align}
&\text{\label{eq:zbarAverage}} \\
&\omega_n = - \sum_{a=1}^{n-1} \omega_a, \quad \text{\tlam}_n = \frac{\sum_{a=1}^{n-1} \omega_a \text{\tilde{z}}_a}{\sum_{a=1}^{n-1} \omega_a}
\end{align}
```

so  $\text{\tilde{z}}_n$  is the  $\omega$ -weighted average of the outgoing  $\text{\tilde{z}}$

We define  $\mathcal{R}_n$  by additionally requiring a unique outlier among the outgoing

```
\begin{align}
&\text{\label{eq:Outlier}} \\
&\text{\tilde{z}}_1 < \text{\tilde{z}}_2 < \cdots < \text{\tilde{z}}_{n-2} < \text{\tilde{z}}_n < \text{\tilde{z}}_{n-1}.
\end{align}
```

Equivalently, among the outgoing labels  $\{1,\dots,n-1\}$ , exactly one label

### Motivating example: $M_{\{12345\}}$ in $\mathcal{R}_{5,4}$

For  $n=5$  the chamber  $\mathcal{R}_{5,4}$  is defined by

```
\begin{align}
\omega_5 < 0, \quad \omega_1, \omega_2, \omega_3, \omega_4 > 0, \quad
\tilde{z}_1 < \tilde{z}_2 < \tilde{z}_3 < \tilde{z}_5 < \tilde{z}_4.
\end{align}
```

Our final result in  $\mathcal{R}_{5,4}$  will be

```
\begin{align}
\label{eq:M12345R54}
\boxed{
M_{12345} \Big|_{\mathcal{R}_{5,4}}
= \Big(\ab{12} + \ab{13} + \ab{14}\Big) \Big(\ab{23} + \ab{24}\Big) \Big)
}
\end{align}
```

This expression will follow from the general all- $n$  formula~\eqref{eq:Pro

\subsection{General formula}

The second main result is the closed form expression for all  $n$  in  $\mathcal{R}_n$

```
\begin{align}
\label{eq:ProductFormula}
\boxed{
M_{1 \cdots n} \Big|_{\mathcal{R}_n}
= \prod_{i=1}^{n-2} \left( \sum_{j=i+1}^{n-1} \ab{ij} \right).
}
\end{align}
```

We now prove~\eqref{eq:ProductFormula} by merging the three steps analogou

\subsection{Vanishing of  $V$  and collapsing the recursion}

We first show that within  $\mathcal{R}_n$  the retarded vertices built only from out  
More precisely, for any set  $S \subseteq \{1, \dots, n-1\}$  with  $|S| \geq 2$ ,

```
\begin{align}
\label{eq:VanishSubset}
V_{\text{tlam}_S} \Big|_{\mathcal{R}_n} = 0.
\end{align}
```

The proof is identical to the causal "largest-time" argument in time-ord  
Write  $\text{tlam}_i = \omega_i(1, \tilde{z}_i)$  with  $\omega_i > 0$  for  $i \in S$ .

For a cut  $A \sqcup B = S$  define  $\Omega_A = \sum_{i \in A} \omega_i > 0$  and weig

```
\begin{align}
[A, B] = \Omega_A \Omega_B (\tilde{z}_A - \tilde{z}_B), \quad
[uv] = \omega_u \omega_v (\tilde{z}_u - \tilde{z}_v).
\end{align}
```

For each fixed tree  $T$  appearing in~\eqref{eq:VertexDef}, at least one ed

```
\begin{align}
(\tilde{z}_{A_e} - \tilde{z}_{B_e})(\tilde{z}_u - \tilde{z}_v) > 0,
\end{align}
```

as follows from the tree weighted-variance identity

```
\begin{align}
\label{eq:TreeVariance}
\sum_{e=(u,v) \in E(T)}
\Omega_{A_e} \Omega_{B_e} (\tilde{z}_{A_e} - \tilde{z}_{B_e})(\tilde{z}_u - \tilde{z}_v)
= \sum_{i < j \in S} \omega_i \omega_j (\tilde{z}_i - \tilde{z}_j)^2 \geq 0.
\end{align}
```

For that edge, the ratio  $\frac{[A_e, B_e]}{[uv]}$  is positive and the corr  
 $\Theta \setminus \frac{[A_e, B_e]}{[uv]}$  vanishes, so  $V_T = 0$  and hence~\eqr







```

\end{align}
This identity is obtained by expanding  $[A_e, B_e]$  into pairwise brackets

\subsection{Tree identity and LSZ reduction}

Define the \emph{Cayley tree factor}
\begin{align}
\label{eq:CayleyFactor}
\mathbb{T}_S &:= \sum_{T \in \mathcal{Trees}(S)} \prod_{(i,j) \in E(T)} \frac{[ij]^2}{p_{ij}} \\
&= \sum_{T \in \mathcal{Trees}(S)} \prod_{(i,j) \in E(T)} \frac{[ij]}{z_{ij} + i0}, \text{sg}
\end{align}
Using \eqref{eq:CayleyIdentity} together with the master identity \eqref{e
\begin{align}
\label{eq:TreeIdentity}
\mathbb{T}_S - \delta_S, V_{\text{tlam}_S} \\
= \frac{1}{P_S^2 + i0} \sum_{\substack{A \sqcup B = S \\ A, B \neq \emptyset}} [A, B]
\end{align}
where  $V_{\text{tlam}_S}$  is the retarded vertex \eqref{eq:VertexDef} evaluated

Solving \eqref{eq:BG} by iterating \eqref{eq:TreeIdentity} yields a partit
Finally, LSZ reduction on the off-shell leg evaluates the on-shell limit o

\bibliographystyle{utphys}
\bibliography{SMGA.bib}

\end{document}

```



awesome. After "Notation and useful identities" and before "Single minus " I was expecting a section on the MHV amplitude. In particular this should define the Cayley tree notation used through the paper, emphasizing the sum over Cayley tree form of MHV, and briefly mention Hodge's determinant. These definitions should help streamline the rest of the paper discussion on single minus. Please output the latex code for such an interlude

Thought for 20m 14s >

<> TeX



```
\subsection{MHV graviton amplitudes and Cayley trees}
```

In gravity there is no color decomposition, so tree amplitudes are fully p  
The  $n$ -point MHV graviton amplitude has two negative-helicity legs; witho  
take them to be  $n-1$  and  $n$ .

With our normalization \eqref{eq:DeltaNormalization}, we write

$$\begin{aligned} & \text{\label{eq:MHVfull}} \\ & \mathcal{M}_n^{\text{MHV}}(1^+, \dots, (n-2)^+, (n-1)^-, n^-) \\ & = i \int \mathcal{M}_n^{\text{MHV}}(\Delta^4) \prod_{k=1}^n p_k, \end{aligned}$$

where  $\mathcal{M}_n^{\text{MHV}}$  is the reduced (or stripped) MHV amplitude.

\paragraph{Cayley tree formula.}

Let  $\mathcal{T}(S)$  denote the set of (unrooted) labeled trees whose v For  $\Gamma \in \mathcal{T}(S)$  we write  $E(\Gamma)$  for its edge set, an Cayley's theorem implies  $|\mathcal{T}(S)| = |S|^{|S|-2}$ .

A permutation-invariant tree formula for the reduced MHV amplitude is then

$$\begin{aligned} & \text{\label{eq:MHVtree1}} \\ & \mathcal{M}_n^{\text{MHV}} \\ & = \frac{1}{\mathcal{V}_{n-1, n}^2} \sum_{\Gamma \in \mathcal{T}(S)} \prod_{(a,b) \in E(\Gamma)} \frac{[ab]}{\mathcal{V}_{ab}} \\ & \prod_{a=1}^{n-2} \mathcal{V}_{a, n-1} \mathcal{V}_{a, n}^{\deg_{\Gamma}(a)-2} \end{aligned}$$

\end{align}

Equivalently, distributing each factor  $\mathcal{V}_{a, n-1} \mathcal{V}_{a, n}$  evenly among

$$\begin{aligned} & \text{\label{eq:MHVtree2}} \\ & \mathcal{M}_n^{\text{MHV}} \\ & = \frac{1}{\mathcal{V}_{n-1, n}^2} \prod_{a=1}^{n-2} \frac{1}{\mathcal{V}_{a, n-1} \mathcal{V}_{a, n}} \\ & \sum_{\Gamma \in \mathcal{T}(S)} \prod_{(a,b) \in E(\Gamma)} \frac{[ab]}{\mathcal{V}_{ab}} \mathcal{V}_{a, n-1} \mathcal{V}_{a, n} \mathcal{V}_{b, n-1} \mathcal{V}_{b, n}. \end{aligned}$$

\end{align}

\paragraph{Hodges determinant.}

An equivalent compact representation is given by Hodges' determinant.

Let  $|x\rangle$  and  $|y\rangle$  be arbitrary reference spinors, and define

$$\begin{aligned} & \text{\label{eq:HodgesMatrix}} \\ & \Phi_{ij} = \\ & \begin{cases} \frac{[ij]}{\mathcal{V}_{ij}}, & i \neq j, \\ \frac{1}{\mathcal{V}_{ik} \mathcal{V}_{kj}} \frac{[ik]}{\mathcal{V}_{ik}} \frac{[kj]}{\mathcal{V}_{kj}}, & i = j. \end{cases} \end{aligned}$$

\end{align}

For any triple  $(i, j, k)$  we also define  $c_{ijk} \equiv \mathcal{V}_{ij} \mathcal{V}_{jk} \mathcal{V}_{ki}$

If  $|\Phi|^{rst}_{ijk}$  denotes the  $(n-3) \times (n-3)$  minor determinant o

$$\begin{aligned} & \text{\label{eq:HodgesDet}} \\ & \mathcal{M}_n^{\text{MHV}} \\ & = (-1)^{n+1} \text{sgn}(ijk, rst) c_{ijk} c_{rst} |\Phi|^{rst}_{ijk}, \end{aligned}$$

where  $\text{sgn}(ijk, rst) = \pm 1$  is the signature of the permutation sending  $(i, j, k)$

The right-hand side is independent of the choices of  $|x\rangle, |y\rangle$ .  
Applying the matrix-tree theorem to [\eqref{eq:HodgesMatrix}](#) yields the Cay

[\paragraph{Regularized Cayley factors.}](#)

The ratios  $[ij]/\langle av_{ij} \rangle$  appearing in [\eqref{eq:MHVtree1}](#) -- [\eqref{eq:MHVt](#)  
In the half-collinear regime where  $\langle av_{ij} \rangle \rightarrow 0$ , the correct distribution

[\begin{align}](#)

[\label{eq:EdgeRegularization}](#)

[\frac{\[ij\]}{\langle av\\_{ij} \rangle}](#)

[=\frac{\[ij\]^2}{p\\_{ij}^2}](#)

[\quad\longrightarrow\quad](#)

[\frac{\[ij\]^2}{p\\_{ij}^2+i0}](#)

[=\frac{\[ij\]}{z\\_{ij}+i0\epsilon, \text{sg}\\_{ij}}\epsilon,](#)

[\end{align}](#)

and we will repeatedly encounter the associated [\emph{Cayley tree factor}](#)

[\begin{align}](#)

[\label{eq:CayleyFactor}](#)

[\mathrm{T}\\_S](#)

[:=\sum\\_{\Gamma \in \text{mathsf{Trees}\(S\)} \prod\\_{\(i,j\) \in E\(\Gamma\)} \frac{\[ij\]}{z\\_{ij}+i0\epsilon, \text{sg}\\_{ij}}\epsilon,](#)

[\end{align}](#)

and its on-shell limits.

These objects play the role in gravity that Parke--Taylor factors play in

